

PSEUDO-CALABI FLOW

XIUXIONG CHEN AND KAI ZHENG

1. ABSTRACT

We first define Pseudo-Calabi flow, as

$$\begin{cases} \frac{\partial \varphi}{\partial t} = -f(\varphi), \\ \Delta_{\varphi} f(\varphi) = S(\varphi) - \underline{S}. \end{cases}$$

Then we prove the well-posedness of this flow including the short time existence, the regularity of the solution and the continuous dependence on the initial data. Next, we point out that the L^{∞} bound on Ricci curvature is an obstruction to the extension of the pseudo-Calabi flow. Finally, we show that if there is a cscK metric in its Kähler class, then for any initial potential in a small $C^{2,\alpha}$ neighborhood of it, the pseudo-Calabi flow must converge exponentially to a nearby cscK metric.

CONTENTS

1. abstract	1
2. Introduction	1
3. Notations and setup	3
4. Energy functionals	6
5. Short time existence of the pseudo-Calabi flow	7
5.1. Continuous dependence on initial data	11
5.2. The main proposition	12
5.3. The technical lemmas	13
5.4. Short time existence of the linearized equation	17
5.5. A priori estimates of the linearized equation	20
5.6. Regularity of the pseudo-Calabi flow	27
6. Long time existence of the pseudo-Calabi flow	29
7. Pseudo-Calabi flow in the space of Kähler metrics	31
7.1. M admits no holomorphic vector field	32
7.2. M admits holomorphic vector fields	33
8. Exponential decay of the pseudo-Calabi flow	37
9. Further remarks and problems	43
References	44

2. INTRODUCTION

A renowned problem in Kähler geometry is to find a constant scalar curvature Kähler metric (cscK) in an arbitrary Kähler class. When restricted to the canonical Kähler class, a cscK metric is nothing but a Kähler-Einstein metric (KE). The cscK

metric satisfies a (totally nonlinear) 4th order partial differential equation of the Kähler potential, and the usual variational method is difficult to apply. Calabi [3][4] suggested to use the heat flow method:

$$\frac{\partial \varphi}{\partial t} = S(\varphi) - \underline{S},$$

which has become known in the literature as the Calabi flow. A critical point of this Calabi flow is precisely a cscK metric. This flow has been actively studied in recent years (c.f.[24][12][18][16][17][54]...). However, the remaining technical difficulties are still daunting simply because it is a 4th order flow.

One wonders if we can take "square root" of the Calabi flow. If so, then it will become a 2nd order flow and we can reduce it to something we are all familiar with. What we propose here is a very natural approach (where we call it Pseudo-Calabi flow):

$$\begin{cases} \frac{\partial \varphi}{\partial t} = -f(\varphi), \\ \Delta_{\varphi} f(\varphi) = S(\varphi) - \underline{S}. \end{cases}$$

In the canonical Kähler class, this flow will reduce to the famous Kähler-Ricci flow. If we start with a metric in general Kähler class, the leading term on the right hand side is the same as the Kähler-Ricci flow (the logarithm of volume ratio between the evolving Kähler metrics and the fixed reference metric). However, there is an additional, unpleasant term, which is 0th order pseudo-differential operator. It is this additional term which makes things very subtle. Nonetheless, it is crucial that the cscK metric is the fixed point of the pseudo-Calabi flow. Moreover, in the space of Kähler metrics equipped with the Calabi gradient metric, the pseudo-Calabi flow is the gradient flow of the K -energy (see Remark 4.1).

We first prove the following theorem:

Theorem 2.1. *If the initial Kähler potential is in $C^{2,\alpha}$ ($0 < \alpha < 1$), then the flow exists for short time. More importantly, it becomes smooth right after $t > 0$.*

Following the corresponding work in the Calabi flow ([18]...), we have

Theorem 2.2. *The L^{∞} bound of the Ricci curvature is the obstruction to an extension of the pseudo-Calabi flow.*

In order to make a case that the pseudo-Calabi flow is the right approach, we need the following stability theorem.

Theorem 2.3. *If there is a cscK metric in its Kähler class, then for any initial potential in a small $C^{2,\alpha}$ neighborhood of it, the pseudo-Calabi flow must converge exponentially to a nearby cscK metric.*

Remark There is an important earlier work in geometric flow which is essential a variant of this flow. In a very interesting work [50], Simanca considered the so called "extremal flow" $\frac{\partial \varphi}{\partial t} = -G_t(S_t - \pi_t S_t) = F(\varphi)$, where π_t is L^2 -orthogonal projection operator onto the space of real holomorphic potentials. In a sense, it is a slight variation of the "Pseudo-Calabi flow", which we consider here, by a lower order term. One of main motivations for such a modification is that fixed points of the extremal flow $S = \pi S$ are precisely extremal metrics. We believe that our

version has a simpler concept and it can also be viewed as a generalization of the Kähler-Ricci flow to the non-canonical Kähler class, since it agrees with the Kähler-Ricci flow in the canonical class. Simanca proved the short time existence of the extremal flow for the G -invariant initial Kähler potentials in the space $C_{(k+1,0)}$ with the norm $\|\varphi\| = \sup_{t \in I} \{ \sup_{0 \leq r \leq k+1} \|\partial_t^r v\|_{W_G^{2(k+1-r),2}} \}$ provided that $2k > n + 2$. First of all, he chose an approximate solution in a small time interval by solving heat equation with the given G -invariant initial data. Secondly, he used semigroup method to obtain the solution v of the linearized equation with the coefficient which is determined by the approximate solution. Thirdly, following Kato's program in [39], he defined a map by solving $\lambda\varphi - F(\varphi) = -v(t) + \lambda(\varphi_0 + \int_0^t v(s)ds)$ for some real number λ , such that $\lambda - F'_{\varphi_0}$ is an isomorphism. Finally, he obtained the fixed point of this map and solved the extremal flow. It is a nice work indeed. However, the proof is difficult to comprehend (from the standard PDE point of view), and our assumption on regularity of initial data is weaker.

Guan [36] defined a modified Ricci flow $\frac{\partial}{\partial t}g = -Ric(g) + HRic(g) + L_V g$ where $HRic(g)$ is the harmonic part of the Ricci form and V is a real holomorphic vector field. Then he considered the problem of finding generalized quasi-Einstein metrics. This flow is another complicated variation of the pseudo-Calabi flow. Guan claimed the short time existence of his flow with $C^\infty(M)$ initial data by a very brief outline.

We remark that, when M admits no holomorphic vector field, these three flows coincide. However, the soliton solutions formed under these three flows are different. We do believe our flow is the simplest which allow us to focus on the main challenges which are arisen from the geometric aspects of cscK metrics.

Acknowledgements: The second author would like to thank Professor W. Y. Ding for constant encouragements over the past several years. He would also like to thank Professor S. B. Angenent for introducing him his paper [1] and Professor J. Y. Li, Professor J. Qing, Professor Y. D. Wang, Professor M. J. Zhu and Professor X. H. Zhu for their helps and encouragements. Thanks also go to ICTP, USTC and UW-Madison for their hospitality during the time when part of the work was done.

We wish to thank Professor S. Simanca for carefully reading an earlier version of this manuscript and for his helpful comments.

3. NOTATIONS AND SETUP

Let M be a n -dimensional compact Kähler manifold. ω is a Kähler form belonging to a fixed Kähler class Ω . In the local coordinates (z_1, z_2, \dots, z_n) , we have

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}.$$

The Riemannian metric corresponding to ω is given by $g = \sum_{i=1}^n g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}$ on $T^\mathbb{C}(M)$. Written in this form, the metric g is Kähler if and only if

$$g_{ij} = g_{\bar{i}\bar{j}} = 0 \text{ and } \frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i}.$$

The volume form is the (n, n) form

$$dV = \omega^{[n]} = \frac{\omega^n}{n!} = \left(\frac{\sqrt{-1}}{2}\right)^n \det(g_{i\bar{j}}) dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^n \wedge dz^{\bar{n}}.$$

For each $\omega \in \Omega$, the corresponding Ricci form

$$Ric = \frac{\sqrt{-1}}{2} \sum_{i=1}^n R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log(\omega^n)$$

is a closed form, in which $\log(\omega^n)$ is generally not a globally defined function on M . The first Chern class is $C_1(M) = \frac{[Ric]}{\pi}$. The scalar curvature is the contraction of the Ricci curvature

$$S = g^{i\bar{j}} R_{i\bar{j}}$$

and the Futaki potential f is the real value solution of

$$\Delta_\varphi f = S - \underline{S}$$

with $\int_M e^f \omega_\varphi^n = V$. Furthermore, since

$$S\omega^n = nRic \wedge \omega^{n-1},$$

we obtain that the average of the scalar curvature is

$$\underline{S} = \frac{\int_M S dV}{V} = \frac{1}{(n-1)!V} \int_M Ric \wedge \omega^{n-1} = \frac{Ric[\omega]^{[n-1]}}{[\omega]^{[n]}} = \frac{\pi C_1(M)[\omega]^{[n-1]}}{[\omega]^{[n]}}$$

which only depends on the Kähler class $[\omega]$. The space of Kähler potentials is defined as

$$\mathcal{H} = \{\varphi \in C^\infty(M, \mathbb{R}) | \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi > 0\}.$$

Donaldson [27], Mabuchi [46] and Semmes [49] defined a Riemannian metric as

$$\int_M f_1 f_2 \omega_\varphi^n$$

for any $f_1, f_2 \in T_\varphi \mathcal{H}$, under which \mathcal{H} becomes a non-positive curved infinite dimensional symmetric space. Chen [11] proved that any two points in \mathcal{H} can be connected by a $C^{1,1}$ geodesics and \mathcal{H} is a metric space. Later, Calabi and Chen proved \mathcal{H} is negatively curved in the sense of Alexanderof in [6]. The space of normarlized Kähler potentials is defined as

$$\mathcal{H}_0 = \{\varphi \in C^\infty(M, \mathbb{R}) | \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi > 0 \text{ and } I(\varphi) = 0\},$$

where

$$(3.1) \quad I(\varphi) = \sum_{p=0}^n \frac{1}{(p+1)!(n-p)!} \int_M \omega^{n-p} (\partial \bar{\partial} \varphi)^p.$$

Calabi [5] suggested another metric on \mathcal{H} ,

$$(3.2) \quad \int_M g_\varphi^{i\bar{j}} f_{1i} f_{2\bar{j}} \omega_\varphi^n$$

for any $f_1, f_2 \in T_\varphi \mathcal{H}_0$. It is computed in Calamai [7] that the Calabi's gradient metric on \mathcal{H}_0 admits a unique Levi-Civita connection.

The pseudo-Calabi flow is defined as

$$\begin{cases} \frac{\partial}{\partial t} g_{i\bar{j}} = -f_{i\bar{j}}, \\ g(0) = g_0, \end{cases}$$

in the fixed but arbitrary Kähler class Ω . According to the definition of the Kähler condition we see that the pseudo-Calabi flow preserves the Kähler condition, i.e.

Theorem 3.1. *If g_0 is Kähler, then $g(t)$ is Kähler if $g(t)$ satisfies the pseudo-Calabi flow.*

The equation for the Kähler form is:

$$(3.3) \quad \begin{cases} \frac{\partial \omega_\varphi}{\partial t} = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} f, \\ \omega_{\varphi(0)} = \omega_0. \end{cases}$$

We observe the following:

Theorem 3.2. *The pseudo-Calabi flow preserves the Kähler class.*

We now show that, when the class Ω is the canonical class, the pseudo-Calabi flow is just the Kähler-Ricci flow. First, we recall that the Kähler-Ricci flow is

$$\frac{\partial g_{\varphi i\bar{j}}}{\partial t} = \lambda g_{\varphi i\bar{j}} - R_{i\bar{j}},$$

where λ is the sign of the first Chern class. Its potential equation is

$$\frac{\partial \varphi}{\partial t} = h + \lambda \varphi - h_\omega,$$

where h_ω , the Ricci potential of the background metric, satisfies

$$Ric(\omega) - \lambda \omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} h_\omega$$

with $\int_M e^{h_\omega} \omega^n = V$ and $\int_M e^{\frac{\partial \varphi}{\partial t} - \lambda \varphi + h_\omega} \omega^n = V$. Next by definition, we have

$$\Delta_\varphi f = S - \underline{S} = -\Delta_\varphi (h + \lambda \varphi - h_\omega).$$

Then the maximum principle implies that $f = -h - \lambda \varphi + h_\omega$. Finally we conclude that $f_{i\bar{j}} = R_{i\bar{j}} - \lambda g_{\varphi i\bar{j}}$, i.e. the pseudo-Calabi flow in the canonical class coincides with the Kähler-Ricci flow.

An observation which is used in the sequel is that the pseudo-Calabi flow can be written as a pseudo-differential Monge-Ampère flow. The potential equation of our flow is

$$(3.4) \quad \begin{cases} \Delta_\varphi \frac{\partial \varphi}{\partial t} = -\Delta_\varphi f = -S_\varphi + \underline{S}, \\ \varphi(0) = \varphi_0. \end{cases}$$

According to [18], we have a decomposition of the scalar curvature as

$$S_\varphi = -\Delta_\varphi h + tr_\varphi Ric(\omega),$$

in which

$$h = \log \frac{\omega_\varphi^n}{\omega^n} = \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})}.$$

Therefore the pseudo-Calabi flow can be rewritten as

$$(3.5) \quad \frac{\partial \varphi}{\partial t} = -f + c(t) = h - P + c(t),$$

where

$$(3.6) \quad \triangle_{\varphi} P = \text{tr}_{\varphi} \text{Ric}(\omega) - \underline{S} = g_{\varphi}^{i\bar{j}} R_{i\bar{j}}(\omega) - \underline{S}$$

under the normalization condition

$$(3.7) \quad \int_M e^P \omega^n = V.$$

The function P is well defined since we have $\int_M \text{tr}_{\varphi} \text{Ric}(\omega) \omega_{\varphi}^n = \underline{S}[\omega_{\varphi}]^{[n]}$. Since the volume is invariant along the pseudo-Calabi flow, we can choose

$$(3.8) \quad \int_M e^{\frac{\partial \varphi}{\partial t} + P} \omega^n = V.$$

such that $c(t) = 0$. Note that for any $c_1(t)$ and $c_2(t)$ defined by two different normalization conditions, the corresponding solutions of (3.5) with the same initial data only differ by the constant $\int_0^t c_1(s) - c_2(s) ds$.

4. ENERGY FUNCTIONALS

In this section, we assume that $\varphi(t)$ is the $C^{\infty}(M, g)$ solution of the pseudo-Calabi flow. According to Theorem 3.2, we see that the flow keeps the volume fixed, so the flow can be viewed as some kind of "normalized" flow. Since the flow preserves the Kähler class and the average of the scalar curvature \underline{S} is an invariant of the Kähler class, \underline{S} stays constant under the flow. The most important observation here is that the K -energy is decreasing along the flow, which makes it reasonable to search for cscK metrics using the pseudo-Calabi flow.

The K -energy is defined by Mabuchi [45] as the following

$$(4.1) \quad \nu_{\omega}(\varphi) = -\frac{1}{V} \int_0^1 \int_M \dot{\varphi}(\tau) (S_{\varphi(\tau)} - \underline{S}) \omega_{\varphi(\tau)}^n d\tau$$

where $\varphi(\tau)$ is a path from 0 to φ .

Theorem 4.1. *The K -energy decreases along the pseudo-Calabi flow. Furthermore, the t -derivative of the K -energy achieves zero at some t if and only if $\omega_{\varphi(t)}$ is a cscK metric.*

Proof. Plugging (3.4) in (4.1), we obtain the derivative of the K -energy along the flow:

$$\delta_{\dot{\varphi}} \nu_{\omega}(\varphi) = -\frac{1}{V} \int_M \dot{\varphi} (S - \underline{S}) \omega_{\varphi}^n = -\frac{1}{V} \int_M |\nabla f|_{g_{\varphi}}^2 \omega_{\varphi}^n.$$

So the K -energy decreases along the flow unless $\nabla f \equiv 0$. Hence $g(t_0)$ is cscK for some t_0 , or $g(t)$ converges to a cscK metric as t tends to infinity if the K -energy is bounded from below. \square

Remark 4.1. Since the derivative of the K -energy at φ is

$$\delta \nu_{\omega}(\varphi) = \frac{1}{V} \int_M g_{\varphi}^{i\bar{j}} \dot{\varphi}_i f_{\bar{j}} \omega_{\varphi}^n,$$

we conclude that (3.5) is the gradient flow of the K -energy in the space \mathcal{H}_0 with the Calabi's gradient metric (3.2).

5. SHORT TIME EXISTENCE OF THE PSEUDO-CALABI FLOW

Let $C^{2,\alpha}(M, g)$ be the completion of the smooth function under the $C^{2,\alpha}$ norm. It is called little Hölder space in customary literature. We shall show that the Cauchy problem for the pseudo-Calabi flow,

$$(5.1) \quad \begin{cases} \frac{\partial}{\partial t} \varphi = h - P, \\ \Delta_{\varphi} P = \text{tr}_{\varphi} \text{Ric}(\omega) - \underline{S}, \\ \varphi(0) = \varphi_0, \end{cases}$$

with the normalization condition (3.7) and (3.8) has a short time solution for any initial Kähler potential in $C^{2,\alpha}(M, g)$.

The proof of local existence with the $C^{2,\alpha}$ initial data is quite different from the case of smooth initial data. DaPrato-Grisvard [25], Angenent [1] and some other mathematicians developed the abstract theory of local existence for the fully nonlinear parabolic equation. They have [25] constructed the continuous interpolation spaces so that the linearized operator stays in certain class.

We apply their ideas to prove the short time existence.

- We first linearize the fully nonlinear equation at the initial data.
- Next we derive a priori estimates, related to our special solution space, and apply it to prove the linearized equation has a local solution by using the contraction mapping theorem.
- Then using Remark 5.3, a key decomposition of the initial data, we derive the energy inequality. Combining the former inequality with the Sobolev imbedding theorem and the bootstrap method, we obtain a priori estimates of the solution to the linearized equation. Thus, the solution exists for all the time and its $C^{2,\alpha}$ norm is continuous in t .
- Finally we construct a sequence of solutions to the linear approximation equations, and show that it is a contractive sequence by choosing small time or small initial data.

Let (x_1, \dots, x_{2n}) be the local real coordinate. We fix a background Kähler metric $g \in C^{\infty}(M, g)$ in $[\omega]$. Let $Q_T = M \times [0, T]$ be the time-space. The point and the distance in Q_T are denoted by $X = (x, t)$ and $d(X, X_0) = (d(x, x_0)^2 + |t - t_0|)^{\frac{1}{2}}$, respectively. The Hölder spaces on Q_T are defined as the following:

$$\begin{aligned} C^{1,\alpha}(Q_T) &= C^{1+\alpha, \frac{1+\alpha}{2}}(Q_T) = \{\varphi \mid |\varphi|_{1+\alpha, \frac{1+\alpha}{2}} = |\varphi|_{C^0(Q_T)} + |D_x \varphi|_{C^0(Q_T)} \\ &\quad + |D_x \varphi|_{C^{\alpha}(Q_T)} + \sup_{x \in M, t \neq s, t, s \in [0, T]} \frac{|\varphi(x, t) - \varphi(x, s)|}{|t - s|^{\frac{1+\alpha}{2}}} < \infty\}, \\ C^{2,\alpha}(Q_T) &= C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T) = \{\varphi \mid |\varphi|_{2+\alpha, 1+\frac{\alpha}{2}} = |\varphi|_{C^0(Q_T)} + |D_x \varphi|_{C^0(Q_T)} \\ &\quad + |D_{xx} \varphi|_{C^{\alpha}(Q_T)} + |D_t \varphi|_{C^{\alpha}(Q_T)}\}. \end{aligned}$$

The Sobolev spaces are defined as

$$\begin{aligned} W^{q,r}(Q_T) &= \{\varphi | [\int_0^T (\int_M |\varphi|^q \omega^n)^{\frac{r}{q}} dt]^{\frac{1}{r}}\}, W^{q,q}(Q_T) = L^q(Q_T), \\ W_2^{2k,k}(Q_T) &= \{\varphi \in L^2(Q_T) \text{ with } \sum_{0 \leq |p|+2q \leq 2k} \|D_x^p D_t^q \varphi\|_{L^2(M)} < \infty\}, \\ W_2^{1,0}(Q_T) &= \{\varphi | \|\varphi\|_{2,Q_T} + \|\nabla \varphi\|_{2,Q_T} < \infty\}, \\ V_2(Q_T) &= \{\varphi \in W_2^{1,0}(Q_T) | \|\varphi\|_{V_2(Q_T)} = \sup_{0 \leq t \leq T} \|\varphi(t)\|_{2;M} + \|\nabla \varphi\|_{2,Q_T} < \infty\}. \end{aligned}$$

We denote

$$X_T^k = C^0([0, T], C^{k+2+\alpha}(M, g)) \cap C^1([0, T], C^{k+\alpha}(M, g))$$

which equipped with the norm $\|\cdot\|_{X_T^k} = \max_{0 \leq t \leq T} [\|\partial_t \cdot\|_{C^{k,\alpha}} + \|\cdot\|_{C^{k+2,\alpha}}]$. We also denote

$$\dot{X}_T^k = C^0([0, T], C^{k+2+\alpha}(M, g)) \cap C^1([0, T], C^{k+\alpha}(M, g)).$$

Here, all the derivatives and norms are defined with respect to the background metric ω . We shall prove the following results.

Theorem 5.1. *Suppose that $\varphi_0 \in C^{2,\alpha}(M, g)$ satisfies $\lambda\omega \leq \omega_{\varphi(0)} \leq \Lambda\omega$ for some positive constants λ and Λ . Then under the normalization conditions (3.7) and (3.8), the Cauchy problem for the pseudo-Calabi flow (5.1) has a unique solution*

$$\varphi(x, t) \in \dot{X}_T^0,$$

where T is the maximal existence time.

Remark 5.1. If in addition $\varphi_0 \in C^{k,\alpha}(M, g)$ we obtain $\varphi(x, t) \in \dot{X}_T^k$.

Theorem 5.2. *Let M admits a cscK metric ω . Suppose that $\varphi_0 \in C^{2,\alpha}(M, g)$ satisfies $\lambda\omega \leq \omega_{\varphi(0)} \leq \Lambda\omega$ for some positive constants λ and Λ . Then for any $T > 0$ there exists a positive constant $\epsilon_0(M, g, T)$. If $|\varphi_0|_{C^{2,\alpha}(M, g)} \leq \epsilon_0$, then the equation has a unique solution on $[0, T]$, and*

$$\|\varphi\|_{X_T^0} \leq C\epsilon_0,$$

where C depends on M, g and T . Furthermore ϵ_0 goes to zero, as T goes to infinity.

Remark 5.2. The imbedding theorem (5.14) implies that the solutions in both theorems satisfy $D_{i\bar{j}}\varphi \in C^{\alpha, \frac{\alpha}{2}}(M \times [0, T])$. Then using the equation (5.1) we obtain $\varphi \in C^{2+\alpha, 1+\frac{\alpha}{2}}(M \times [0, T])$.

Chen's conjecture [14] says that a global $C^{1,1}$ K -energy minimizer in any Kähler class must be smooth. This conjecture has been proved in the canonical Kähler class via the weak Kähler-Ricci flow [15][22][23][51]. We hope that the pseudo-Calabi flow will be the right approach to solve this conjecture. In Subsection 5.5, we obtain a partial estimates related to this conjecture.

Remark 5.3. For any $\varphi_0 \in C^{2,\alpha}$, we can choose a smooth function $\bar{\phi}$ which approximates φ_0 in $C^{2,\alpha}$ norm. Let $\phi = \bar{\phi} - \int_M \bar{\phi} \omega^n$. Then we replace the reference metric

ω and φ by $\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi$ and $\tilde{\varphi}_0 = \varphi_0 - \phi$ respectively in the equation (5.1), so that

$$\begin{cases} \frac{\partial \tilde{\varphi}}{\partial t} = \log \frac{\det(\tilde{g}_{i\bar{j}} + \tilde{\varphi}_{i\bar{j}})}{\det(\tilde{g}_{i\bar{j}})} - P(\tilde{\varphi}), \\ \Delta_{\tilde{\varphi}} P(\tilde{\varphi}) = g_{\tilde{\varphi}}^{i\bar{j}} R_{i\bar{j}}(\omega) - \underline{S}, \\ \tilde{\varphi}(0) = \tilde{\varphi}_0 = \varphi_0 - \phi \in C^{2,\alpha}(M). \end{cases}$$

It is obvious that $\tilde{\varphi} + \phi$ gives the solution to the original equation. Here $|\tilde{\varphi}_0|_{C^{2,\alpha}(M)}$ could be small enough to be used later. Moreover, we have $\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi > 0$, since $\omega_{\varphi_0} - \tilde{\omega} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\tilde{\varphi}_0$ is sufficiently small.

Proof. (proof of Theorem 5.1) To prove the theorem we employ the idea of the inverse function theorem in [2][37] and its adaption to a parabolic equation in [1][25].

We introduce the following notations

$$\begin{aligned} J &= [0, T], E_1 = C^{2,\alpha}(M), E_0 = C^\alpha(M), \\ X &= \{\varphi \in C^1(J, E_0) \cap C^0(J, E_1) \mid \|\varphi\|_X \\ &= \max_{[0, T]} (\|\partial_t \varphi\|_{C^\alpha(M)} + \|\varphi\|_{C^{2,\alpha}(M)}) < \infty\}, \\ Y &= \{\psi = (\varphi_1, \varphi_2) \in C^0(J, E_0) \times E_1 \mid \|\psi\|_Y \\ &= \max_{[0, T]} \|\varphi_1\|_{C^\alpha(M)} + \|\varphi_2\|_{C^{2,\alpha}(M)} < \infty\}. \end{aligned}$$

Let $U = \{\varphi \in E_1 \mid \omega_\varphi > 0\}$. Then U is an open subset of E_1 and $\varphi_0 \in U$. We define the map Φ from U to Y as the following

$$(5.2) \quad \begin{aligned} \Phi : C^0(J, U) \cap C^1(J, E_0) &\rightarrow C^0(J, E_0) \times U, \\ \varphi &\mapsto (\partial_t \varphi - h + P, \varphi_0). \end{aligned}$$

Note that the pseudo term $P(\varphi)$ belongs to $C^0(J, E_0)$ by Lemma 5.6 which we will prove below. Moreover, Lemma 5.6 assures Φ is infinite Fréchet differentiable at any point of U . Choosing the approximate solution

$$(5.3) \quad \bar{\varphi}(t) \equiv \varphi_0 \text{ on } [0, T],$$

we get

$$(5.4) \quad \Phi(\bar{\varphi}) = (-h(\varphi_0) + P(\varphi_0), \varphi_0) \triangleq (\bar{\psi}, \varphi_0) \triangleq \psi_0.$$

Now we compute the linearized equation in a usual way. Let the variation of φ be

$$\dot{\varphi} \triangleq \frac{\partial}{\partial s}(\varphi + sv)|_{s=0} = v.$$

Differentiating both sides of (3.6), we get

$$\dot{\Delta}_\varphi P + \Delta_\varphi \dot{P} = -v^{i\bar{j}} R_{i\bar{j}}(\omega).$$

Thus we obtain

$$\Delta_\varphi \dot{P} = v^{i\bar{j}} [P_{i\bar{j}} - R_{i\bar{j}}(\omega)] = -v^{i\bar{j}} T_{i\bar{j}}.$$

Here the $(1, 1)$ -tensor

$$(5.5) \quad T_{i\bar{j}} = -P_{i\bar{j}} + R_{i\bar{j}}(\omega)$$

is a harmonic tensor which is smooth when $\varphi \in C^3$. Since Φ is at least C^1 Fréchet differentiable in U , we get

$$\begin{aligned}\Phi'(\bar{\varphi}) : C^0(J, E_1) &\rightarrow C^0(J, E_0) \times E_1, \\ v &\mapsto \left(\frac{\partial v}{\partial t} - \Delta_{\bar{\varphi}} v - Q, v_0\right).\end{aligned}$$

Here the function Q satisfies the following equation

$$(5.6) \quad \Delta_{\bar{\varphi}} Q = v^{i\bar{j}} T_{i\bar{j}} = -v^{i\bar{j}} (P_{i\bar{j}} - R_{i\bar{j}}(\omega))$$

with the normalization condition

$$(5.7) \quad \int_M Q e^{P(\bar{\varphi})} \omega^n = 0$$

by differentiating (3.7). Then Q belongs to $C^0(J, E_0)$, according to Lemma 5.6. By using the following Proposition 5.4 for special case $\bar{\varphi} = \varphi_0$, we deduce that the linearized operator $\Phi'(\bar{\varphi})$ is a linear isomorphism.

In order to seek a unique solution φ satisfying the flow equation $\Phi(\varphi) = (0, \varphi_0)$, firstly we introduce the closed ball of $C^0(J, U)$ by

$$B(0, \epsilon) = \{\rho \in C^0(J, U) \mid \max_{[0, T]} \|\rho\|_{C^{2, \alpha}(M)} \leq \epsilon\}$$

for some ϵ determined later. Then for any $\rho \in B(0, \epsilon)$ we define a map A by

$$(5.8) \quad A\rho = [\Phi'(\bar{\varphi})]^{-1}[\psi - \Phi(\bar{\varphi} + \rho)] + \rho.$$

Letting $\rho_0 = 0$, we obtain a sequence by $\rho_{n+1} = A\rho_n$ inductively. Secondly, we show that ρ_n is a contractive sequence. For any ρ_1 and ρ_2 in $B(0, \epsilon)$, we compute

$$A\rho_1 - A\rho_2 = [\Phi'(\bar{\varphi})]^{-1} \left[\int_0^1 [\Phi'(\bar{\varphi} + s\rho_2 + (1-s)\rho_1) - \Phi'(\bar{\varphi})](\rho_2 - \rho_1) ds \right].$$

Since Φ is a C^1 map on open subset U , there exists a small constant ϵ such that

$$\|\Phi'(\bar{\varphi} + s\rho_2 + (1-s)\rho_1) - \Phi'(\bar{\varphi})\| \leq \frac{1}{2\|[\Phi'(\bar{\varphi})]^{-1}\|}.$$

Consequently, we obtain

$$(5.9) \quad \max_{[0, T]} \|A\rho_1 - A\rho_2\|_{C^{2, \alpha}} \leq \frac{1}{2} \max_{[0, T]} \|\rho_2 - \rho_1\|_{C^{2, \alpha}}.$$

Moreover, for all $T' \leq T$, ϵ is uniform since $\|[\Phi'(\bar{\varphi})]^{-1}\| \geq \frac{1}{\|[\Phi'(\bar{\varphi})]\|}$ is uniformly bounded below. Thirdly, we verify that A maps $B(0, \epsilon)$ into itself. By the triangle inequality and (5.9) we have

$$\begin{aligned}\max_{[0, T]} \|A\rho\|_{C^{2, \alpha}} &\leq \max_{[0, T]} \|A\rho - A\rho_0\|_{C^{2, \alpha}} + \max_{[0, T]} \|A\rho_0\|_{C^{2, \alpha}} \\ &\leq \frac{1}{2}\epsilon + \max_{[0, T]} \|A(0)\|_{C^{2, \alpha}}.\end{aligned}$$

To estimate the second term, we use (5.4) and (5.11) to obtain

$$A(0) = [\Phi'(\bar{\varphi})]^{-1}[(0, \varphi_0) - \Phi(\bar{\varphi})] = [\Phi'(\bar{\varphi})]^{-1}(h(\varphi_0) - P(\varphi_0), 0) \in X.$$

According to Proposition 5.4, $A(0)$ solves the following equations,

$$\begin{cases} \frac{\partial \bar{v}}{\partial t} - \Delta_{\bar{\varphi}} \bar{v} - Q(\bar{v}) = h(\varphi_0) - P(\varphi_0) \text{ in } Q_T, \\ \Delta_{\bar{\varphi}} Q(\bar{v}) = \bar{v}^{i\bar{j}} T_{i\bar{j}} \text{ in } Q_T, \\ \bar{v}(0) = 0. \end{cases}$$

Since \bar{v} is continuous in t by Lemma 5.33, one could choose a time T_2 small enough such that

$$(5.10) \quad \max_{[0, T_2]} \|\bar{v}\|_{C^{2,\alpha}(M)} \leq \frac{\varepsilon}{2}.$$

Note that $T_2 \leq T$, after we replace the time T by T_2 in all of the argument above, both the spaces X , Y , and the mappings Φ and $\Phi'(\bar{\varphi})$ may change. However the result in Proposition 5.4 and the contraction property of the mapping A remain valid. Meanwhile, since the injectivity of the linearized operator implies $[\Phi'(\bar{\varphi})]_{T_2}^{-1}[\psi - \Phi(\bar{\varphi})|_{T_2}] = \bar{v}|_{T_2}$, the following inequity holds

$$\max_{[0, T_2]} \|A(0)\|_{C^{2,\alpha}(M)} = \max_{[0, T_2]} \|[\Phi'(\bar{\varphi})]_{T_2}^{-1}[\psi - \Phi(\bar{\varphi})|_{T_2}]\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}\varepsilon.$$

Hence, A is a contractive mapping and it maps $B(0, \varepsilon)$ into itself. So the contraction mapping theorem implies that there do exists a fixed point $\rho \in B(0, \varepsilon)$ such that $\psi = \Phi(\varphi_0 + \rho) \in C^0(J, U)$ is a solution of (5.1) on the small lifespan $[0, T_2]$. Furthermore, we deduce $\varphi_0 + \rho \in U$ from the equation (5.1). Therefore the main Theorem 5.1 follows by solving the flow equation with the initial data, given by the value of the solution at the end of the previous interval, till the maximal existence time T . \square

Remark 5.4. In fact, we can produce the approximate solution by using the following Monge-Ampère type flow

$$(5.11) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{\omega_{\varphi}^n}{\omega^n} - P(\varphi_0), \\ \varphi(0) = \varphi_0, \end{cases}$$

since Cao [8] proved the long time existence of such flow.

Proof. (proof of Theorem 5.2) In order to solve $\Phi(\varphi) = (0, \varphi_0)$, we only need to verify the following inequality

$$\max_{[0, T]} \|A(0)\|_{C^{2,\alpha}(M)} = \max_{[0, T]} \|[\Phi'(0)]^{-1}(0, \varphi_0)\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}\varepsilon.$$

So it suffices to choose φ_0 such that $|\varphi_0|_{C^{2,\alpha}}$ is a small constant depending on T . Thus, the rest of the proof of Theorem 5.2 follows along the same lines of the rest of the proof of Theorem 5.1. \square

5.1. Continuous dependence on initial data. The continuous dependence on initial data of solutions is used to study the stability of the pseudo-Calabi flow near a cscK metric which is the equilibrium solution of this flow.

Theorem 5.3. *If ϕ is a solution of (5.1) for initial data ϕ_0 on $[0, T]$, then there is a neighborhood U of ϕ_0 such that (5.1) has a solution $\varphi(t)$ on $[0, T]$ for any $\varphi_0 \in U$ and the mapping $\varphi_0 \mapsto \varphi(t)$ is C^k for $k = 0, 1, 2, \dots$*

Proof. We derive, substituting $\psi = \varphi - \varphi_0$ into the potential equation (5.1),

$$\begin{cases} \frac{\partial}{\partial t} \psi = \log \frac{\omega_{\psi+\varphi_0}^n}{\omega^n} - P(\psi + \varphi_0), \\ \Delta_{\psi+\varphi_0} P(\psi + \varphi_0) = \text{tr}_{\psi+\varphi_0} \text{Ric}(\omega) - \underline{S}, \\ \psi(0) = 0. \end{cases}$$

It is obvious that its solution plus φ_0 gives the solution of (5.1). Analogously to the mapping (5.2), we define a mapping as

$$\begin{aligned} \Phi : C^1(J, E_0) \cap C^0(J, U) \times U &\rightarrow C^0(J, E_0), \\ (\psi, \varphi_0) &\mapsto \partial_t \psi - \log \frac{\omega_{\psi+\varphi_0}^n}{\omega^n} + P(\psi + \varphi_0). \end{aligned}$$

Then we have $\Phi(\phi - \phi_0, \phi_0) = 0$ and the Fréchet derivative with respect to ψ at $(\phi - \phi_0, \phi_0)$ is given by

$$\begin{aligned} \Phi'(\phi - \phi_0, \phi_0) : C^1(J, E_0) \cap C^0(J, E_1) \times E_1 &\rightarrow C^0(J, E_0), \\ (v, w) &\mapsto \frac{\partial v}{\partial t} - \Delta_\phi v - Q(\phi). \end{aligned}$$

We apply Remark 5.3 and Proposition 5.4 with $\varphi = \phi$. Hence $\Phi'(\phi - \phi_0, \phi_0)$ is an isomorphism and the theorem follows from the implicit function theorem. \square

5.2. The main proposition. From here on we shall simply write φ for both approximate solutions $\bar{\varphi}$ and ϕ . We are going to prove the following proposition for φ ; we stress that φ may depend on t . Due to Remark 5.3, we can further assume that φ satisfies

$$(5.12) \quad \max_{[0, T]} |\varphi(t)|_{C^{2, \alpha}} \leq \delta \ll 1$$

for some δ to be determined later. Generally, we introduce the space

$$(5.13) \quad V_\delta = \{\psi \in C^0([0, T], U) \mid \max_{[0, T]} |\psi(t)|_{C^{2, \alpha}} \leq \delta\}.$$

Proposition 5.4. *Under the normalization condition (5.7) the linearized equation*

$$(5.14) \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta_\varphi v - Q = u(x, t) \text{ in } Q_T, \\ \Delta_\varphi Q(v) = v^{i\bar{j}}(R_{i\bar{j}}(\omega) - P_{i\bar{j}}) \text{ in } Q_T, \\ v(0) = w, \end{cases}$$

has a unique solution $v(x, t) \in X$, for any $u(x, t) \in C^0([0, T], C^\alpha(M))$ and $w \in C^{2, \alpha}(M)$.

Since the linearized operator is not self-adjoint, we can not use the Fredholm theory directly. In order to prove the Proposition 5.4, we first show that the linearized equation has a short time solution by using the contraction mapping theorem. Then we prove that the solution of the linearized equation exists for any t . Before demonstrating the procedure, we need the following technical lemmas.

5.3. The technical lemmas. We first deal with the pseudo-differential term P which satisfies (3.6) and (3.7). The following identity is computed directly.

Lemma 5.5. *Suppose that φ_1 and φ_2 are two $C^{2,\alpha}$ Kähler potentials. Let $g_a = ag_{\varphi_2} + (1-a)g_{\varphi_1}$. We have*

$$(5.15) \quad \Delta_{\varphi_1}(P(\varphi_1) - P(\varphi_2)) = - \int_0^1 g_a^{i\bar{l}} g_a^{k\bar{j}} d(\varphi_1 - \varphi_2)_{k\bar{l}} (R_{i\bar{j}} - P(\varphi_2)_{i\bar{j}}).$$

The regularity of P in the space direction can be improved due to the regularity of φ .

Lemma 5.6. *For any $\varphi(x, t) \in C^0([0, T], U)$, we have*

$$P \in C^0([0, T], C^{2+\alpha}(M))$$

and

$$T_{i\bar{j}} = -P_{i\bar{j}} + R_{i\bar{j}}(\omega) \in C^0([0, T], C^\alpha(M)).$$

Proof. Since $\varphi(x, t) \in C^0([0, T], U)$, we have

$$\text{tr}_\varphi \text{Ric}(\omega) - \underline{S} \in C^0([0, T], C^\alpha(M)).$$

It follows directly from the Schauder estimate that

$$P \in L^\infty([0, T], C^{2+\alpha}(M)).$$

Let

$$g_{\varphi(t_s)} = sg_{\varphi(t_1)} + (1-s)g_{\varphi(t_2)}.$$

Then we plug $\varphi_i = \varphi(t_i)$ for $i = 1, 2$ in (5.15) to get

$$\Delta_{\varphi(t_1)}(P(t_1) - P(t_2)) = - \int_0^1 g_{\varphi(t_s)}^{k\bar{j}} g_{\varphi(t_s)}^{i\bar{l}} ds (\varphi(t_1) - \varphi(t_2))_{k\bar{l}} [R_{i\bar{j}}(\omega) - P(t_2)_{i\bar{j}}].$$

Freezing the coefficient by the fixed metric ω , we get

$$(5.16) \quad \begin{aligned} \Delta(P(t_1) - P(t_2)) &= (\Delta - \Delta_{\varphi(t_1)})(P(t_1) - P(t_2)) \\ &\quad - \int_0^1 g_{\varphi(t_s)}^{k\bar{j}} g_{\varphi(t_s)}^{i\bar{l}} ds (\varphi(t_1) - \varphi(t_2))_{k\bar{l}} [R_{i\bar{j}}(\omega) - P(t_2)_{i\bar{j}}]. \end{aligned}$$

Let $f = P(t_1) - P(t_2)$. Then the Green representation gives

$$f - \int_M f \omega^n = -\frac{1}{V} \int_M \Delta f \cdot G(g)(x, y) \omega^n(y)$$

which implies

$$(5.17) \quad |f - \int_M f \omega^n| \leq \delta |D^2 f| + C |D^2(\varphi(t_1) - \varphi(t_2))|$$

by condition (5.12). Since the normalization condition (3.7) implies both $\sup_M P$ and $-\inf_M P$ are nonnegative, we infer that

$$\begin{aligned} 0 &\leq \sup_M f \leq \int_M f \omega^n + \delta |D^2 f| + C |D^2(\varphi(t_1) - \varphi(t_2))|, \\ 0 &\geq \inf_M f \geq \int_M f \omega^n - \delta |D^2 f| - C |D^2(\varphi(t_1) - \varphi(t_2))|. \end{aligned}$$

Then it follows that

$$-\delta |D^2 f| - C |D^2(\varphi(t_1) - \varphi(t_2))| \leq \int_M f \omega^n \leq \delta |D^2 f| + C |D^2(\varphi(t_1) - \varphi(t_2))|.$$

So by (5.17) we get

$$|f| \leq 2\delta|D^2f| + C|D^2(\varphi(t_1) - \varphi(t_2))|.$$

By applying the Schauder estimate to (5.16) again, we deduce that

$$|f|_{C^{2,\alpha}(M)} \leq C(\delta|D^2f| + |D^2(\varphi(t_1) - \varphi(t_2))| + |D^2(\varphi(t_1) - \varphi(t_2))|_{C^\alpha(M)}).$$

Let δ be small enough so that

$$|f|_{C^{2,\alpha}(M)} \leq C(|D^2(\varphi(t_1) - \varphi(t_2))| + |D^2(\varphi(t_1) - \varphi(t_2))|_{C^\alpha(M)}).$$

Then $|P(t)|_{C^{2,\alpha}}$ is continuous with respect to t , since $\varphi \in C^0([0, T], U)$. \square

Now we construct a sequence of smooth Kähler potentials approximating φ in $C^0([0, T], C^{2,\alpha}(M))$.

Lemma 5.7. *For any $C^{2,\alpha}(M)$ Kähler potential φ_0 , there exists a sequence of smooth Kähler potentials φ_n such that*

$$\lim_{n \rightarrow \infty} |\varphi_n - \varphi_0|_{C^{2,\alpha}(M)} = 0.$$

Moreover, the corresponding harmonic tensors

$$\lim_{n \rightarrow \infty} |T_{ni\bar{j}} - T_{0i\bar{j}}|_{C^\alpha(M)} = 0.$$

Proof. Let $\phi_0 = \Delta\varphi_0$. We select a sequence of smooth functions ϕ_n such that it converges to ϕ_0 in $C^\alpha(M)$. Let φ_n be the smooth solution of $\Delta\varphi_n = \phi_n$ with the normalization condition $\int_M \varphi_n \omega^n = 0$. The Schauder estimate implies

$$|\varphi_n - \varphi_0|_{C^{2,\alpha}(M)} \leq C(|\phi_n - \phi_0|_{C^\alpha(M)} + |\varphi_n - \varphi_0|_{C^0(M)}).$$

Meanwhile, the Green representation gives the following L^∞ bound of φ_n

$$|\varphi_n - \varphi_0|_{C^0(M)} \leq C(|\phi_n - \phi_0|_{C^0(M)}).$$

Then by combining these two estimates, we have ω_{φ_n} is a Kähler form and the first part of the lemma follows. The second part of the lemma follows from Lemma 5.6 by replacing $P(t_1)$ and $P(t_2)$ with $P(\varphi_n)$ and $P(\varphi_0)$ respectively in (5.16). \square

Lemma 5.8. *For any $\varphi(x, t) \in C^0([0, T], U)$, there exists a sequence of smooth Kähler potentials φ_n such that*

$$\lim_{n \rightarrow \infty} \max_{[0, T]} |\varphi_n - \varphi_0|_{C^{2,\alpha}(M)} = 0.$$

Moreover, the corresponding harmonic tensors satisfy

$$\lim_{n \rightarrow \infty} \max_{[0, T]} |T_{ni\bar{j}} - T_{0i\bar{j}}|_{C^\alpha(M)} = 0.$$

Proof. Similar to Lemma 5.7 $\varphi_n \in C^\infty(M \times [0, T])$ is obtained. The rest part of the lemma follows from Lemma 5.6. \square

For lacking of maximal principle, the following lemmas play an important role in solving the linearized equation Q satisfying (5.6) and (5.7). Let $H_0^p(M, g_\varphi) = \{\eta \in H^p(M, g_\varphi) | \int_M \eta \omega_\varphi^n = 0\}$. Analogously to P , the function Q is characterized by the following lemmas.

Lemma 5.9. *Suppose that φ_1 and φ_2 are two $C^{2,\alpha}$ Kähler potentials. Let $g_a = ag_{\varphi_2} + (1-a)g_{\varphi_1}$. We have for any C^2 functions v_1 and v_2*

$$\begin{aligned} \Delta_{\varphi_1}(Q(v_1, \varphi_1) - Q(v_2, \varphi_2)) &= (-v_1^{i\bar{j}} + v_2^{i\bar{j}})(P_{i\bar{j}}(\varphi_2) - R_{i\bar{j}}(\omega)) \\ &\quad - v_1^{i\bar{j}}(P(\varphi_1) - P(\varphi_2))_{i\bar{j}} - \int_0^1 g_a^{i\bar{l}} g_a^{k\bar{j}} (\varphi_2 - \varphi_1)_{k\bar{l}} Q(\varphi_2)_{i\bar{j}} da. \end{aligned}$$

Similarly to Lemma 5.6, we have

Lemma 5.10. *For any $\varphi(x, t) \in C^0([0, T], U)$ and any $v \in C^0([0, T], C^{2+\alpha}(M))$, we have*

$$Q \in C^0([0, T], C^{2+\alpha}(M)).$$

The following lemma enables us to derive the L^p bound of Q .

Lemma 5.11. *Suppose that $\varphi \in U$. For any $\eta \in H_0^0(M, g_\varphi)$, let ρ be the solution of $\Delta_\varphi \rho = \eta$ given by the L^p theory. Then we have*

$$\|\rho^{i\bar{j}}\|_{2;M;g_\varphi} = \|\eta\|_{2;M;g_\varphi},$$

and for all $p > 1$, there exists

$$\|\rho^{i\bar{j}}\|_{p;M;g_\varphi} \leq C \|\eta\|_{p;M;g_\varphi}.$$

Proof. We compute for $p = 2$

$$\|\rho^{i\bar{j}}\|_{2;M;g_\varphi}^2 = \int_M |\Delta_\varphi \rho|^2 \omega_\varphi^n = \|\eta\|_{2;M;g_\varphi}^2.$$

For $p > 2$ the lemma follows from the L^p estimate. \square

Lemma 5.12. *Suppose that $\varphi \in U$. If $v \in L^2(M)$, then there exists a weak solution $Q \in L^2(M)$ of (5.6) in the distribution sense, for any $\zeta \in C^\infty(M)$*

$$\int_M \Delta_\varphi \zeta Q \omega_\varphi^n = \int_M v T_{i\bar{j}} \zeta^{i\bar{j}} \omega_\varphi^n.$$

Moreover, for any $\eta \in L^2(M)$

$$\int_M \eta Q \omega_\varphi^n \leq |T|_{0;Q_T} \|v\|_{2;M} \|\eta\|_{2;M}.$$

Proof. According to Lemma 5.8 there exist three sequences of smooth φ_n , $T_{ni\bar{j}}$ and v_n such that $\varphi_n \rightarrow \varphi$ in $C^0([0, T], C^{2,\alpha})$, $T_{ni\bar{j}} \rightarrow T_{i\bar{j}}$ in $C^0([0, T], C^\alpha)$ and $v_n \rightarrow v$ in $L^2(M)$. Let Q_n be the smooth solution of the equation

$$\Delta_\varphi Q_n = v_n^{i\bar{j}} T_{ni\bar{j}} = (v_n T_{ni\bar{j}})^{i\bar{j}}$$

and ρ be the $W^{2,2}(M)$ solution of

$$\Delta_\varphi \rho = \eta - \frac{1}{V} \int_M \eta \omega_\varphi^n.$$

Note that we have

$$\int_M \eta(Q_n - \frac{1}{V} \int_M Q_n \omega_\varphi^n) \omega_\varphi^n = \int_M \rho (v_n T_{ni\bar{j}})^{i\bar{j}} \omega_\varphi^n = \int_M \rho^{i\bar{j}} v_n T_{ni\bar{j}} \omega_\varphi^n.$$

Then by invoking Lemma 5.11 and the Hölder inequality, we have

$$\begin{aligned} \int_M \eta(Q_n - \frac{1}{V} \int_M Q_n \omega_\varphi^n) \omega_\varphi^n &\leq \|\rho^{i\bar{j}}\|_{2;M,g_\varphi} \|v_n T_{n i \bar{j}}\|_{2;M,g_\varphi} \\ &\leq C |T_n|_{0;Q_T} \|\eta\|_{2;M} \|v_n\|_{2;M}. \end{aligned}$$

It follows that $Q_n - \frac{1}{V} \int_M Q_n \omega_\varphi^n$ weakly converges to some function Q_1 in $L^2(M)$. Furthermore, since for any $\zeta \in C^\infty(M)$

$$\int_M \Delta_\varphi \zeta Q_n \omega_\varphi^n = \int_M v_n T_{n i \bar{j}} \zeta^{i\bar{j}} \omega_\varphi^n,$$

after taking the limit we obtain the desired weak solution $Q = Q_1 - \frac{\int_M Q_1 e^{P(\varphi)} \omega^n}{\int_M e^{P(\varphi)} \omega^n} \in L^2(M)$, whence the lemma follows. \square

By using the L^p estimate instead of the L^2 estimate, we can deduce that

Lemma 5.13. *Suppose that $\varphi \in C^0([0, T], U)$. If $v \in L^p(Q_T)$, then there exists a weak solution $Q \in L^p(Q_T)$ of (5.6) in the distribution sense, for any $\zeta \in C^\infty(Q_T)$*

$$\int_{Q_T} \Delta_\varphi \zeta Q \omega_\varphi^n dt = \int_{Q_T} v T_{i \bar{j}} \zeta^{i\bar{j}} \omega_\varphi^n dt.$$

Moreover, for any $\eta \in L^q(Q_T)$ with $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_{Q_T} \eta Q \omega_\varphi^n \leq C |T|_{0;Q_T} \|v\|_{p;Q_T} \|\eta\|_{q;Q_T}.$$

Now we generalize a priori estimates for parabolic equation in a bounded domain to an arbitrary Riemannian manifold M . We introduce some function spaces first. Consider the following space

$$B([0, t_0], C^{k,\alpha}(M)) = \{v(t) \in C^0(M), \forall t \in [0, t_0] \mid \sup_{[0, t_0]} \|v\|_{C^{k,\alpha}} < \infty\}.$$

Define also

$$C^{\alpha,0}(M \times [0, t_0]) = \{v \in C^0(M \times [0, t_0]) \mid v(t, \cdot) \in C^\alpha(M), \forall t \in [0, t_0]\};$$

equip this space with the norm

$$\|v\|_{C^{\alpha,0}(M \times [0, t_0])} = \sup_{0 \leq t \leq t_0} \|v(t)\|_{C^{2,\alpha}(M)}.$$

Finally consider the space

$$C^{2+\alpha,1}(M \times [0, t_0]) = \{v \in C^{2,1}(M \times [0, t_0]) \mid D_t v, D_{i\bar{j}} v \in C^\alpha(M), \forall t \in [0, t_0]\}$$

endowed with the norm

$$\|v\|_{C^{2+\alpha,1}(M \times [0, t_0])} = \|v\|_\infty + \sum_{i=1}^n \|D_i v\|_\infty + \|D_t v\|_{C^{\alpha,0}} + \sum_{i,j=1}^n \|D_{i\bar{j}} v\|_{C^{\alpha,0}}.$$

We cite an imbedding Lemma 5.1.1 in Lunardi's book [44].

Lemma 5.14. *(Lunardi [44]) If $v \in C^{2+\alpha,1}(M \times [0, t_0])$, then $D_{i\bar{j}} v \in C^{\alpha, \frac{\alpha}{2}}(M \times [0, t_0])$.*

The following optimal regularity theorem in a domain Ω is a part of Theorem 5.1.13 in Lunardi's book [44] for a linear parabolic equation

$$(5.18) \quad u_t = a^{i\bar{j}} u_{i\bar{j}} + b^i u_i + cu + f.$$

Theorem 5.15. (Lunardi [44]) *Let Ω be an open set in R^n of uniformly $C^{2,\alpha}$ boundary. Let $f \in UC(\bar{\Omega} \times [0, t_0]) \cap C^{\alpha,0}(\bar{\Omega} \times [0, t_0])$ be such that $f(t, x) = 0$ for every $t \in [0, t_0]$ and $x \in \partial\Omega$. Suppose that the coefficients a^{ij} , b^i and c belong to $C^\alpha(\bar{\Omega})$ with $0 < \alpha < 1$, and a^{ij} satisfies $\Lambda|\xi|^2 \geq a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 > 0$ for any $\xi \in R^n \setminus \{0\}$. If $u_0 \in C^0(\bar{\Omega})$, then (5.18) has a unique solution which belongs to $C([0, T] \times \Omega)$. For every $\epsilon \in (0, t_0]$ there is C such that*

$$(5.19) \quad \|u\|_{C^{2+\alpha,1}(\bar{\Omega} \times [\epsilon, t_0])} \leq \frac{C}{\epsilon^{\frac{\alpha}{2}+1}} (\|u_0\|_{L^\infty(\bar{\Omega})} + \|f\|_{C^{\alpha,0}(\bar{\Omega} \times [0, t_0])}).$$

If also $u_0 \in C^{2,\alpha}(\bar{\Omega})$, then there exists C' such that

$$(5.20) \quad \|u\|_{C^{2+\alpha,1}(\bar{\Omega} \times [0, t_0])} \leq C' (\|u_0\|_{C^{2,\alpha}(\bar{\Omega})} + \|f\|_{C^{\alpha,0}(\bar{\Omega} \times [0, t_0])}).$$

Since M is compact, it can be covered by a finite number of charts $\{\Omega_p\}$. Let $\eta_p(t) \in C_0^\infty(\Omega_p)$ be a partition of unity subordinate to the charts $\{\Omega_p\}$. We assume $u_0 \in C^{2,\alpha}(\bar{\Omega})$ and consider the following equation for $\eta_p u$

$$\begin{aligned} & (\eta_p u)_t - a^{ij}(\eta_p u)_{ij} - b^i(\eta_p u)_i - c\eta_p u \\ & = \tilde{f} = \eta_p f - 2a^{ij}[(\eta_p)_i u]_j - a^{ij}(\eta_p)_{ij} u + b^i u(\eta_p)_i \text{ in } \Omega_p. \end{aligned}$$

Since $\tilde{f}|_{\partial\Omega_p} = 0$, applying Theorem 5.15 with $\Omega = \Omega_p$, we have

$$\|\eta_p u\|_{C^{2+\alpha,1}(\bar{\Omega}_p \times [0, t_0])} \leq C(\|u_0\|_{L^\infty(M)} + \|\tilde{f}\|_{C^{\alpha,0}(\bar{\Omega}_p \times [0, t_0])}).$$

Combining all estimates in each ball and using the Hölder inequality, we obtain

$$(5.21) \quad \|u\|_{C^{2+\alpha,1}(M \times [0, t_0])} \leq C(\|u\|_{C^0(M \times [0, t_0])} + \|u_0\|_{C^{2,\alpha}(M)} + \|f\|_{C^{\alpha,0}(M \times [0, t_0])}).$$

For any $f \in C^{\alpha,0}(M \times [\epsilon, t_0])$ we can choose a smooth sequence f_i which converges to f in $C^{\alpha,0}(M \times [\epsilon, t_0])$. Then by the classical existence theory (see [42]), one has a unique solution $u_i \in C^{2+\alpha,1+\frac{\alpha}{2}}$ of (5.18) for each f_i . The maximum principle implies that $\|u\|_{C(M \times [0, t_0])}$ has a uniform bound. Consequently, one can derive the solution of (5.18) from the compactness of u_i . The uniqueness of the solution also follows from the maximum principle. In conclusion, we obtain:

Theorem 5.16. *Let $f \in UC(M \times [0, t_0]) \cap C^{\alpha,0}(M \times [0, t_0])$ and $u_0 \in C^{2,\alpha}(M)$. If the coefficients a^{ij} , b^i and c belong to $C^\alpha(M)$ with $0 < \alpha < 1$, and a^{ij} satisfies $\Lambda|\xi|^2 \geq a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 > 0$ for any $\xi \in R^n \setminus \{0\}$. Then (5.18) has a unique solution $u \in C^{2+\alpha,1}(M \times [0, t_0])$ on M , and*

$$\|u\|_{C^{2+\alpha,1}(M \times [0, t_0])} \leq C(\|u_0\|_{C^{2,\alpha}(M)} + \|f\|_{C^{\alpha,0}(M \times [0, t_0])}).$$

5.4. Short time existence of the linearized equation. We now prove the short time existence of the linearized equation by using the contraction mapping theorem. Since the complex Laplacian is a real operator, we prove all a priori estimates in the real coordinates.

Since the coefficients of the leading terms depending on t create some problems, we freeze these coefficients and analyze the corresponding modified equation instead. This approach is similar to [32] but more complicated, because of the pseudo-differential term in our equations.

Before discussing the proof we recall the definition and some features of the heat kernel on a compact Riemannian manifold (M, g) in [9][48][26][34]. Suppose that

$\rho(\lambda) \in C^\infty([0, \infty))$ satisfies

$$\begin{cases} \rho(\lambda) = 1, \lambda < \frac{\text{inj}(M)}{4}, \\ \rho(\lambda) = 0, \lambda > \frac{\text{inj}(M)}{2}. \end{cases}$$

Set $\rho(x, y) = \rho(d(x, y))$. On M there exists a complete orthonormal basis $\{f_k\}$ of $L^2(M)$, consisting of the eigenfunctions f_k of Δ with eigenvalues λ_k . Then the heat kernel is given by

$$H(x, y, t) = \sum_{k=0}^{\infty} e^{-\lambda_k t} f_k(x) f_k(y).$$

Proposition 5.17. ([9][48][26][34]) *As $t \rightarrow 0$, the heat kernel has an asymptotic expansion*

$$H(x, y, t) = \frac{1}{(2\sqrt{\pi})^n} t^{-\frac{n}{2}} \exp^{-\frac{d^2(x, y)}{4t}} \left(\rho \sum_{k=0}^K t^k \phi_k + O(t^{K+1}) \right),$$

where $K > \frac{n}{2} + 2$. The expression is independent of K , and ϕ_k are fixed functions constructed by Minakshisundaram-Pleijel. The heat kernel has the properties:

- (i) $(\frac{\partial}{\partial t} - \Delta_x)H(x, y, t) = 0$.
- (ii) $H(x, y, t)$ is smooth except $x = y$, positive and symmetric in x and y .
- (iii) $\int_M H(x, y, t) \omega^n(y) = 1$ for any $t > 0$ and any $x \in M$.
- (iv) $\lim_{t \rightarrow 0} \int_M H(x, y, t) f(y) \omega^n(y) = f(x)$ for any $f \in C^0(M)$.
- (v) $|H(x, y, t)| \leq Ct^{-\frac{n}{2}} e^{-\frac{d^2(x, y)}{t}}$,
 $|\nabla_x H(x, y, t)| \leq Ct^{-\frac{(n+1)}{2}} e^{-\frac{d^2(x, y)}{t}}$.
 In particular, these inequalities imply
 $|H(x, y, t)| \leq Ct^{-\beta} d^{-n+2\beta}$,
 $|\nabla_x H(x, y, t)| \leq Ct^{-\beta} d^{-n-1+2\beta}$, for some constant C and any $\beta \in (0, 1)$.
- (vi) The solution of $\frac{\partial}{\partial t} \varphi - \Delta \varphi = f \in C^0(M \times (0, t_0])$ with $\varphi(t = 0) = \varphi(0) \in C^0(M)$ is of the form

$$\varphi(x, t) = \int_M H(x, y, t) \varphi(y, 0) \omega^n(y) + \int_0^t \int_M H(x, y, t-s) f(y, s) \omega^n(y) ds$$

on $(0, t_0] \times M$.

Consider the modified equation

$$(5.22) \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta v + \Delta v - \Delta_\varphi v - Q = u(x, t) \text{ in } Q_T, \\ \Delta_\varphi Q = v^{i\bar{j}} T_{i\bar{j}} \text{ in } Q_T, \\ v(0) = w. \end{cases}$$

Let $\tilde{v} = v - w$. Then the solution of the original linearized equation is obtained, by adding w to the solution of the following equations

$$\begin{cases} \frac{\partial \tilde{v}}{\partial t} - \Delta \tilde{v} + \Delta \tilde{v} - \Delta_\varphi \tilde{v} - \Delta_\varphi w - \tilde{Q} - W = u(x, t) \text{ in } Q_T, \\ \Delta_\varphi \tilde{Q} = \tilde{v}^{i\bar{j}} T_{i\bar{j}} \text{ in } Q_T, \\ \Delta_\varphi W = w^{i\bar{j}} T_{i\bar{j}} \text{ in } Q_T, \\ \tilde{v}(0) = 0. \end{cases}$$

So we can solve (5.22) with $w = 0$ instead. Introduce the function space

$$Z = \{v \in C^{2+\alpha,0}(M \times [0, t_0]) | v(0) = 0\}$$

with the norm $\|v\| = \sup_{0 \leq t \leq t_0} |v(t)|_{C^{2,\alpha}(M)}$. We define the following operator:

$$(5.23) \quad \hat{v} = T(v) = \int_0^t \int_M H(x, y, t-s) \{Q + u - \Delta v + \Delta_\varphi v\}(y, s) \omega^n(y) ds.$$

Let $\tilde{f} = \int_0^t \int_M H(x, y, t-s) f(y, s) \omega^n(y) ds$.

Lemma 5.18. *If $f \in C^0(M \times [0, t_0])$, then*

$$|\tilde{f}|_{C^0(M)} \leq C t_0^{1-\beta} \sup_{0 \leq t \leq t_0} |f|_{C^0(M)}.$$

Proof. From Proposition 5.17 (v), it follows directly

$$\begin{aligned} & \left| \int_0^t \int_M H(x, y, t-s) f(y, s) \omega^n(y) ds \right|_{C^0(M)} \\ & \leq \int_0^t \int_M H(x, y, t-s) \omega^n ds \sup_{0 \leq t \leq t_0} |f|_{C^0(M)} \\ & \leq C t_0^{1-\beta} \sup_{0 \leq t \leq t_0} |f|_{C^0(M)}. \end{aligned}$$

□

Lemma 5.19. *T is a map from Z to itself.*

Proof. Since $v \in Z$, the Schauder estimate implies $Q \in C^{2+\alpha,0}(M \times [0, t_0])$. Hence, we obtain that Tv belongs to $C^{2+\alpha,1}(M \times [0, t_0])$ and $Tv(0) = 0$ from Theorem 5.16. □

Lemma 5.20. *T is a contraction map.*

Proof. We compute for any $v_1, v_2 \in Z$

$$\begin{aligned} |T(v_1) - T(v_2)|_{C^{2,\alpha}(M)} &= \left| \int_0^t \int_M [H(x, y, t-s) \{Q(v_1) - Q(v_2) \right. \\ & \quad \left. + (\Delta_\varphi - \Delta)(v_1 - v_2)\}(y, s)] \omega^n(y) ds \right|_{C^{2,\alpha}(M)}. \end{aligned}$$

According to Lemma 5.18 and Theorem 5.16, we obtain

$$\begin{aligned} |T(v_1) - T(v_2)|_{C^{2,\alpha}(M)} &\leq C_1 t_0^{1-\beta} \sup_{0 \leq s \leq t_0} |Q(v_1) - Q(v_2)|_{C^{2,\alpha}(M)} \\ &\quad + C_2 \sup_{0 \leq s \leq t_0} |(\Delta_\varphi - \Delta)(v_1 - v_2)|_{C^\alpha(M)}. \end{aligned}$$

Since $Q(v_1) - Q(v_2)$ satisfies

$$\Delta_\varphi[Q(v_1) - Q(v_2)] = (v_1 - v_2)^{i\bar{j}} T_{i\bar{j}},$$

by applying Lemma 5.6 we have that the first term can not exceed

$$C_3 t_0^{1-\beta} \sup_{0 \leq s \leq t_0} |v_1 - v_2|_{C^{2,\alpha}(M)}.$$

By using the following identity

$$(\Delta_\varphi - \Delta)(v_1 - v_2) = - \int_0^1 g_{s\varphi}^{k\bar{j}} g_{s\varphi}^{i\bar{l}} ds \varphi_{i\bar{j}} (v_1 - v_2)_{k\bar{l}}$$

and the condition (5.12), we see that the second term can be estimated as

$$|(\Delta_\varphi - \Delta)(v_1 - v_2)|_{C^\alpha(M)} \leq C_4 \delta |v_1 - v_2|_{C^{2,\alpha}(M)}.$$

Here δ becomes smaller while t_0 goes to 0. Adding all these estimates, we have

$$|T(v_1) - T(v_2)|_{C^{2,\alpha}(M)} \leq (C_3 t_0^{1-\beta} + C_5 \delta) \sup_{0 \leq s \leq t_0} |v_1 - v_2|_{C^{2,\alpha}(M)}.$$

Accordingly if t_0 and δ are small enough, T is a contraction with constant $\frac{1}{2}$. \square

Therefore we get a fixed point v that satisfies $Tv = v \in Z$ by the contraction mapping theorem. Lastly, we check that the fixed point is the solution of (5.22). Obviously, there holds

$$v = T(v) = \int_0^t \int_M H(x, y, t-s) \{Q + u - \Delta v + \Delta_\varphi v\}(y, s) \omega^n(y) ds.$$

So we get $v \in C^{2+\alpha,1}(M \times [0, t_0])$. Then after differentiating on the both sides, we obtain that v satisfies (5.14) on $[0, t_0] \times M$ on account of Proposition 5.17 (vi). As a result, we conclude that

Theorem 5.21. *The linearized equations (5.14) have a local solution $v \in C^{2+\alpha,1}(M \times [0, t_0])$ under the normalization condition (5.7).*

Remark 5.5. In Kruřkov-Kastro-Lopes [40], they proved the following. Suppose $\Omega = R^n$, $f = f_1 + f_2$, $f_1 \in B([0, T], C^\beta)$, $f_2 \in B([0, T], C^\alpha)$, $\alpha > \beta$ and $u(t, x) \in B([0, T], C^{2,\beta})$ is a solution of (5.18); then $u \in C^{0,2+\beta}$ and

$$(5.24) \quad \sup_{[0,t]} |u|_{C^{2,\beta}} \leq M [\sup_{[0,T]} |f_1|_{C^{2,\beta}} + \int_0^t (t-\tau)^{\frac{\alpha-\beta}{2}-1} \sup_{[0,\tau]} |f_2|_{C^{2,\beta}} d\tau]$$

is satisfied for any $0 \leq t \leq T$. Note that in our linearized equation, $u - \Delta v + \Delta_\varphi v \in C^0([0, T], C^\alpha)$ holds and $Q \in C^0([0, T], C^\gamma)$ for any γ satisfying $1 > \gamma \geq \alpha$ follows from Lemma 5.10; so their estimate is sufficient in our proof.

In the next subsection, we are going to prove an a priori estimate of the solution of the linearized equation to make sure that the short time solution of this linearized equation can be extended to any fixed time t under the condition (5.12) on φ .

5.5. A priori estimates of the linearized equation. Suppose v is the solution of the linearized equations (5.14) under the normalization condition (5.7). Meanwhile, suppose that $\varphi \in V_\delta$ (see (5.13)) satisfies

$$(5.25) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} - \tilde{P}, \\ \varphi(0) = \varphi_0. \end{cases}$$

Here \tilde{P} belongs to $C^0([0, T], C^{2+\alpha}(M))$. In this section, we will prove the following extension theorem.

Theorem 5.22. *For any $T > 0$, the equation (5.14) with (5.7) has a solution $v \in C^{2+\alpha,1}(Q_T)$.*

Since our potential φ belongs to $C^0([0, T], U)$, Lemma 5.8 implies that there is a sequence $\varphi_n \in C^\infty(Q_T)$ such that

$$\varphi_n \rightarrow \varphi \in C^0([0, T], U).$$

In the following lemmas, since the constants in all inequalities do not contain any derivatives of φ more than $\max_{[0,T]} |\partial_t \varphi|_{C^\alpha(M)} + \max_{[0,T]} |\varphi|_{C^{2,\alpha}(M)}$, we omit the approximation process for convenience.

Lemma 5.23. *If v satisfies the following heat equation with the initial data φ_0*

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta_\varphi v = u(x, t) \text{ in } Q_T, \\ v(0) = w, \end{cases}$$

then we have

$$\|\Delta_\varphi v\|_{2;Q_T,g_\varphi}^2 \leq C(\|v\|_{2;Q_T,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2 + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2),$$

where C depends on $n, Q_T, \lambda_0, \Lambda_0$ and the moduli of continuity of $g_\varphi^{i\bar{j}}$ on M .

Proof. Since $\tilde{v} = v - w$ satisfies the following equation with zero initial data

$$\frac{\partial \tilde{v}}{\partial t} - \Delta_\varphi \tilde{v} = u(x, t) + \Delta_\varphi w \text{ in } Q_T,$$

by the L^2 estimate we obtain

$$\|\Delta_\varphi \tilde{v}\|_{2;Q_T,g_\varphi}^2 \leq C(\|\tilde{v}\|_{2;Q_T,g_\varphi}^2 + \|u + \Delta_\varphi w\|_{2;Q_T,g_\varphi}^2).$$

□

Lemma 5.24. *There exists a constant C depending on \tilde{P} such that*

$$(5.26) \quad \frac{1}{2} \int_M |v|^2 \partial_t \omega_\varphi^n \leq \delta \|\nabla_\varphi v\|_{2;M,g_\varphi}^2 + C\|v\|_{2;M,g_\varphi}^2 + \delta^2 \|\Delta_\varphi v\|_{2;M,g_\varphi}^2.$$

Proof. Using (5.25), we express the left hand side of the inequality in the statement as

$$\begin{aligned} \frac{1}{2} \int_M |v|^2 \partial_t \omega_\varphi^n &= \frac{1}{2} \int_M |v|^2 \Delta_\varphi (\partial_t \varphi) \omega_\varphi^n \\ &= \frac{1}{2} \int_M |v|^2 \Delta_\varphi \left(\log \frac{\omega_\varphi^n}{\omega^n} - \tilde{P} \right) \omega_\varphi^n \\ &= \int_M |\nabla_\varphi v|^2 \log \frac{\omega_\varphi^n}{\omega^n} + \int_M v \Delta_\varphi v \log \frac{\omega_\varphi^n}{\omega^n} - \frac{1}{2} \int_M |v|^2 \Delta_\varphi \tilde{P} \omega_\varphi^n. \end{aligned}$$

Therefore the lemma follows from the condition (5.12) and the Cauchy-Schwarz inequality. □

Lemma 5.25. *If v is the solution of (5.14), then we have*

$$(5.27) \quad \begin{aligned} \frac{1}{2} \int_{Q_T} |v|^2 \partial_t \omega_\varphi^n &\leq \delta \|\nabla_\varphi v\|_{2;Q_T,g_\varphi}^2 \\ &+ C(\|v\|_{2;Q_T,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2 + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2) \end{aligned}$$

where C depends on $n, Q_T, \lambda_0, \Lambda_0, \tilde{P}, |T|_{C^0(Q_T)}$ and the moduli of continuity of $g_\varphi^{i\bar{j}}$.

Proof. By using (5.25), we calculate

$$\begin{aligned}
& \frac{1}{2} \int_{Q_T} |v|^2 \partial_t \omega_\varphi^n \wedge dt = \frac{1}{2} \int_{Q_T} |v|^2 \Delta_\varphi (\partial_t \varphi) \omega_\varphi^n \wedge dt \\
& = \frac{1}{2} \int_{Q_T} |v|^2 \Delta_\varphi (\log \frac{\omega_\varphi^n}{\omega^n} - \tilde{P}) \omega_\varphi^n \wedge dt \\
& = \int_{Q_T} |\nabla_\varphi v|^2 \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \wedge dt + \int_{Q_T} v \Delta_\varphi v \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \wedge dt \\
& \quad - \frac{1}{2} \int_{Q_T} |v|^2 \Delta_\varphi \tilde{P} \omega_\varphi^n \wedge dt.
\end{aligned}$$

Using the condition (5.12) and the Cauchy-Schwarz inequality we get

$$\frac{1}{2} \int_{Q_T} |v|^2 \partial_t \omega_\varphi^n \wedge dt \leq \delta \|\nabla_\varphi v\|_{2;Q_T,g_\varphi}^2 + C \|v\|_{2;Q_T,g_\varphi}^2 + \delta^2 \|\Delta_\varphi v\|_{2;Q_T,g_\varphi}^2.$$

By applying Lemma 5.23 to (5.14), we obtain that the right-hand side can be bounded by terms

$$\begin{aligned}
& \delta \|\nabla_\varphi v\|_{2;Q_T,g_\varphi}^2 + C \|v\|_{2;Q_T,g_\varphi}^2 \\
& + \delta^2 C (\|v\|_{2;Q_T,g_\varphi}^2 + \|u + Q\|_{2;Q_T,g_\varphi}^2 + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2).
\end{aligned}$$

Furthermore, Lemma 5.13 gives the bound of the above

$$\delta \|\nabla_\varphi v\|_{2;Q_T,g_\varphi}^2 + C (\|v\|_{2;Q_T,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2 + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2).$$

Therefore the lemma follows. \square

Lemma 5.26. *If v is the solution of (5.14), then we have*

$$(5.28) \quad \sup_{0 \leq t \leq T} \|v\|_{2;M,g_\varphi} \leq C (\|u\|_{L^2(Q_T,g_\varphi)} + \|w\|_{2;M,g_\varphi} + \|\Delta_\varphi w\|_{2;M,g_\varphi}),$$

where C depends on $n, Q_T, \lambda_0, \Lambda_0, \tilde{P}, |T|_{C^0(Q_T)}$ and the moduli of continuity of $g_\varphi^{i\bar{j}}$.

Proof. By using (5.14) we calculate

$$\begin{aligned}
& \partial_t \frac{1}{2} \int_M |v|^2 \omega_\varphi^n = \int_M v \partial_t v \omega_\varphi^n + \frac{1}{2} \int_M |v|^2 \Delta_\varphi (\partial_t \varphi) \omega_\varphi^n \\
& = \int_M v (\Delta_\varphi v + Q + u) \omega_\varphi^n + \frac{1}{2} \int_M |v|^2 \Delta_\varphi (\partial_t \varphi) \omega_\varphi^n.
\end{aligned}$$

Applying Lemma 5.12 and Lemma 5.24, we get

$$\begin{aligned}
\partial_t \frac{1}{2} \int_M |v|^2 \omega_\varphi^n & \leq -\|\nabla_\varphi v\|_{M,g_\varphi}^2 + C \|v\|_{2;M,g_\varphi}^2 + \|u\|_{2;M,g_\varphi}^2 \\
& + \delta \|\nabla_\varphi v\|_{M,g_\varphi}^2 + C \|v\|_{2;M,g_\varphi}^2 + \delta^2 \|\Delta_\varphi v\|_{2;M,g_\varphi}^2 \\
& \leq C(t) \|v\|_{2;M,g_\varphi}^2 + \|u\|_{2;M,g_\varphi}^2 + \delta^2 \|\Delta_\varphi v\|_{2;M,g_\varphi}^2.
\end{aligned}$$

Then the Gronwall's inequality implies

$$\begin{aligned}
\frac{1}{2} \|v\|_{2;M,g_\varphi}^2 &\leq e^{\int_0^t C(s)ds} \left[\frac{1}{2} \|v(0)\|_{2;M,g_\varphi}^2 + \int_0^t \|u\|_{2;M,g_\varphi}^2 + \delta^2 \|\Delta_\varphi v\|_{2;M,g_\varphi}^2 ds \right] \\
&\leq C \left[\frac{1}{2} \|w\|_{2;M,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2 + \delta^2 \|\Delta_\varphi v\|_{2;Q_T,g_\varphi}^2 \right] \\
&\leq C \left[\frac{1}{2} \|w\|_{2;M,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2 \right. \\
&\quad \left. + \delta^2 C (\|v\|_{2;Q_T,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2 + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2) \right].
\end{aligned}$$

The last inequality follows from Lemma 5.23. Therefore (5.28) holds, if we choose δ small enough so that $\delta^2 C \leq \frac{1}{4}$. \square

Lemma 5.27. *If v is the solution of (5.14), then we have*

$$(5.29) \quad \|\nabla_\varphi v\|_{2;Q_T,g_\varphi}^2 \leq C (\|u\|_{2;Q_T,g_\varphi}^2 + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2)$$

where C depends on $n, Q_T, \lambda_0, \Lambda_0, \tilde{P}, |T|_{C^0(Q_T)}$ and the moduli of continuity of $g_\varphi^{i\bar{j}}$.

Proof. Using (5.14) we have

$$\begin{aligned}
\|\nabla_\varphi v\|_{2;Q_T}^2 &= - \int_0^T \int_M v \Delta_\varphi v \omega_\varphi^n \wedge dt = \int_{Q_T} \left\{ -v \frac{\partial v}{\partial t} + v Q \right\} \omega_\varphi^n \wedge dt + \int_{Q_T} v u \omega_\varphi^n \wedge dt \\
&= -\frac{1}{2} \left[\int_{Q_T} \partial_t (|v|^2 \omega_\varphi^n) \wedge dt - \int_{Q_T} |v|^2 \partial_t \omega_\varphi^n \wedge dt \right] + \int_{Q_T} v(Q+u) \omega_\varphi^n \wedge dt.
\end{aligned}$$

Then applying Lemma 5.13 and Lemma 5.25 we have

$$\begin{aligned}
\|\nabla_\varphi v\|_{2;Q_T}^2 &\leq \frac{1}{2} \|w\|_{2;M,g_\varphi}^2 + \delta \|\nabla_\varphi v\|_{2;Q_T,g_\varphi}^2 + C (\|v\|_{2;Q_T,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2 \\
&\quad + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2) + C (\|v\|_{2;Q_T,g_\varphi}^2 + \|u\|_{2;Q_T,g_\varphi}^2).
\end{aligned}$$

Consequently, we get to the following estimate

$$\|\nabla_\varphi v\|_{2;Q_T,g_\varphi}^2 \leq C (\|u\|_{2;Q_T,g_\varphi}^2 + \|w\|_{2;M,g_\varphi}^2 + \|\Delta_\varphi w\|_{2;M,g_\varphi}^2),$$

provided that $\delta \leq \frac{1}{2}$. \square

Proposition 5.28. *(Energy inequality) If v is the solution of (5.14), then we have that v is uniformly bounded in $V_2(Q_T)$;*

$$\|v\|_{V_2(Q_T)} = \sup_{0 \leq t \leq T} \|v\|_{2;M} + \|\nabla v\|_{2;Q_T} \leq C [\|w\|_{2;M} + \|\Delta_\varphi w\|_{2;M} + \|u\|_{2;Q_T}],$$

where C depends on $n, Q_T, \lambda_0, \Lambda_0, \tilde{P}, |T|_{C^0(Q_T)}$ and the moduli of continuity of $g_\varphi^{i\bar{j}}$.

Proof. One can triangulate g and g_φ at the same time in the normal coordinates such that $g_{i\bar{j}} = \delta_{ij}$ and $g_{\varphi i\bar{j}} = \delta_{ij} + \varphi_{i\bar{i}}$. Then we get

$$\frac{1}{\Lambda_0^2} w_{i\bar{k}} w_{k\bar{i}} \leq g_\varphi^{i\bar{j}} g_\varphi^{k\bar{l}} w_{i\bar{l}} w_{k\bar{j}} = \frac{1}{1 + \varphi_{i\bar{i}}} \frac{1}{1 + \varphi_{k\bar{k}}} w_{i\bar{k}} w_{k\bar{i}} \leq \frac{1}{\lambda_0^2} w_{i\bar{k}} w_{k\bar{i}},$$

since $\lambda g \leq g_\varphi \leq \Lambda g$. Accordingly,

$$\|\Delta_\varphi w\|_{2;M,g_\varphi}^2 = \int_M g_\varphi^{i\bar{j}} g_\varphi^{k\bar{l}} w_{i\bar{l}} w_{k\bar{j}} \omega_\varphi^n$$

is equivalent to $\|\Delta_\varphi w\|_{2;M,g}^2$. Analogously, $\|\nabla v\|_{2;M,g_\varphi}$ and $\|v\|_{2;M,g_\varphi}$ are equivalent to $\|\nabla v\|_{2;M,g}$ and $\|v\|_{2;M,g}$ respectively. As a result we combine (5.28) and (5.29) with the conclusion above to obtain the energy inequality. \square

We introduce the following imbedding theorem ([42] Page 77). Let $m = 2n$ be the real dimension.

Theorem 5.29. (*Ladyženskaja, Solonnikov and Ural'ceva* [42]) *If $v \in V_2(Q_T)$, one has the imbedding estimate*

$$|v|_{W^{q,r}(Q_T)} \leq C|v|_{V_2(Q_T)}$$

for

$$C = 2\beta(r, n, q) + T^{\frac{1}{r}} V^{-\frac{1}{2} + \frac{1}{q}}$$

and q, r satisfying the relation $\frac{1}{r} + \frac{m}{2q} = \frac{m}{4}$ with

$$\begin{cases} r \in [2, \infty], q \in [2, \frac{2m}{m-2}] \text{ for } m > 2; \\ r \in [2, \infty], q \in [2, \infty] \text{ for } m = 2. \end{cases}$$

The Sobolev imbedding theorem states:

Theorem 5.30. (*Ladyženskaja, Solonnikov and Ural'ceva* [42]) *Suppose that $v \in W_p^{2k,k}(Q_T)$, $p \geq 1$, $m \geq 2$ and $0 \leq r + 2s = \mu < 2k$.*

If $(2k - \mu)p < m + 2$, then $D_t^s D_x^r v \in L^q(Q_T)$ for $q = \frac{(m+2)p}{m+2-(2k-\mu)p}$;
if $(2k - \mu)p > m + 2$, then $D_t^s D_x^r v \in C^\alpha(Q_T)$ for $\alpha = (2k - \mu) - \frac{m+2}{p}$.

These imbedding theorems and the energy inequality imply the following estimate by the standard bootstrap method.

Lemma 5.31. *There exists a positive constant C such that*

$$|v|_{C^{1+\alpha, \frac{1}{2}+\frac{\alpha}{2}}(Q_T)} \leq C(\|w\|_{W^{2,p}(M)} + \|u\|_{p;Q_T}).$$

Proof. Owing to the energy inequality, Proposition 5.28, we have $v \in V_2(Q_T)$. Then the imbedding theorem, Theorem 5.29, implies a uniform $L^p(Q_T)$ bound for $p = \frac{2(m+2)}{m}$ i.e.

$$\|v\|_{L^p(Q_T)} \leq C\|v\|_{V_2(Q_T)}.$$

According to Lemma 5.13 we obtain that Q has uniform $L^p(Q_T)$ bound, i.e.

$$\|Q\|_{L^p(Q_T)} \leq C\|v\|_{L^p(Q_T)}.$$

Then the parabolic L^p theory tells us v has uniform $W_p^{2,1}(Q_T)$ bound; i.e.

$$\begin{aligned} \|v\|_{W_p^{2,1}(Q_T)} &\leq C(\|v\|_{L^p(Q_T)} + \|Q\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)} + \|w\|_{W^{2,p}(M)}) \\ &\leq C(\|v\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)} + \|w\|_{W^{2,p}(M)}). \end{aligned}$$

The Sobolev imbedding theorem, Theorem 5.30, implies

$$v \in L^{p_1}(Q_T)$$

for $p_1 = \frac{(m+2)p}{m+2-2p} = \frac{2(m+2)}{m-4} > p$.

Analogously, we get $Q \in L^{p_1}(Q_T)$ and $v \in W_{p_1}^{2,1}(Q_T)$. Repeating the similar process, we derive $p_k = \frac{2(m+2)}{m-4k} > m + 2$ for some step k . Then according to the Sobolev imbedding theorem, Theorem 5.30, we have

$$v \in C^{1+\alpha, \frac{1+\alpha}{2}}(Q_T).$$

Thus the lemma follows. \square

Proposition 5.32. *There exists a positive constant C such that*

$$|v|_{C^{2+\alpha,1}(Q_T)} \leq C(|w|_{C^{2,\alpha}(M)} + \sup_{[0,T]} |u|_{C^\alpha(M)}).$$

Proof. Because of Lemma 5.8 one can select a sequence $\varphi_n \in C^\infty(Q_T)$ and $T_{ni\bar{j}} \in C^\infty(Q_T)$ such that $\varphi_n \rightarrow \varphi \in C^0([0, T], C^{2,\alpha})$ and $T_{ni\bar{j}} \rightarrow T_{i\bar{j}} \in C^0([0, T], C^\alpha)$. We omit the standard approximation argument in the sequel of the proof. Since M is compact, M can be covered by finite number of balls $\{B_p(2r)\}$ with radii $2r$. Let $0 \leq \eta_p \leq 1$ be a smooth partition of unity subordinate to the covering $\{B_p(2r)\}$. Since $T_{i\bar{j}}$ is a harmonic tensor, using (5.6) we have

$$\Delta_\varphi Q = [v^i T_{i\bar{j}}]^{\bar{j}}.$$

Now on each ball $B(2r) \in \{B_p(2r)\}$, we calculate

$$(5.30) \quad \Delta_\varphi(\eta Q) = [\eta v^i T_{i\bar{j}}]^{\bar{j}} - \eta^{\bar{j}} v^i T_{i\bar{j}} + \Delta_\varphi \eta Q + 2\nabla_\varphi \eta \nabla_\varphi Q.$$

Then the Schauder estimate for the first derivatives in [35] (Corollary 8.35) provides us the following estimate

$$\begin{aligned} \sup_{[0,T]} |\eta Q|_{C^{1,\alpha}(B(r))} &\leq C(\sup_{[0,T]} |Q|_{C^0(B(2r))} + \sup_{[0,T]} |v^i T_{i\bar{j}}|_{C^\alpha(B(2r))}) \\ &\quad + \sup_{[0,T]} |-\eta^{\bar{j}} v^i T_{i\bar{j}} + \Delta_\varphi \eta Q + 2\nabla_\varphi \eta \nabla_\varphi Q|_{C^0(B(2r))}. \end{aligned}$$

Combining these estimates on each covering and using the interpolation inequality for the Hölder space, we obtain

$$(5.31) \quad \sup_{[0,T]} |Q|_{C^{1,\alpha}(M)} \leq C(\sup_{[0,T]} |Q|_{0;M} + \sup_{[0,T]} |v|_{C^{1,\alpha}(M)}).$$

Since ηv satisfies (5.30) on the ball $B(2r)$, the L^∞ bound of Q follows from Theorem 8.16 in [35], for some $q > n$

$$\begin{aligned} \sup_{B_{2r}} |\eta Q| &\leq C(\|\eta v^i T_{i\bar{j}}\|_{L^q(B_{2r})} + \|-\eta^{\bar{j}} v^i T_{i\bar{j}} + \Delta_\varphi \eta Q + 2\nabla_\varphi \eta \nabla_\varphi Q\|_{L^{\frac{q}{2}}(B_{2r})}) \\ &\leq C(|v|_{C^1(B_{2r})} + \|Q\|_{L^{\frac{q}{2}}(B_{2r})} + \|\nabla Q\|_{L^{\frac{q}{2}}(B_{2r})}). \end{aligned}$$

Note that the interpolation inequality for the Sobolev space implies

$$\|\nabla Q\|_{L^{\frac{q}{2}}(B_{2r})} \leq \epsilon \|Q\|_{W^{2,\frac{q}{2}}(B_{2r})} + C(\epsilon) \|Q\|_{L^{\frac{q}{2}}(B_{2r})};$$

moreover, the L^p estimate of (5.6) gives that

$$\|Q\|_{W^{2,\frac{q}{2}}(B_{2r})} \leq C(\|Q\|_{L^{\frac{q}{2}}(M)} + |v|_{C^2(M)}).$$

Combining the above estimates, we infer that

$$(5.32) \quad \max_M |Q| \leq \sum_q \max_{B_q(2r)} |\eta_q Q| \leq C(|v|_{C^1(M)} + \|Q\|_{L^{\frac{q}{2}}(M)} + \epsilon |v|_{C^2(M)}).$$

Plugging Lemma 5.31 and (5.31) in Theorem 5.16 we obtain

$$\begin{aligned}
& \sup_{[0,T]} |\partial_t v|_{C^\alpha(M)} + \sup_{[0,T]} |v|_{C^{2,\alpha}(M)} \\
& \leq C(\sup_{[0,T]} |Q|_{C^\alpha(M)} + |w|_{C^{2,\alpha}(M)} + \sup_{[0,T]} |u|_{C^\alpha(M)}) \\
& \leq C(\sup_{[0,T]} |Q|_{0;M} + \sup_{[0,T]} |v|_{C^{1,\alpha}(M)} + |w|_{C^{2,\alpha}(M)} + \sup_{[0,T]} |u|_{C^\alpha(M)}) \\
& \leq C(\sup_{[0,T]} |Q|_{0;M} + |w|_{C^{2,\alpha}(M)} + \sup_{[0,T]} |u|_{C^\alpha(M)}).
\end{aligned}$$

Then (5.32) gives the bound of the right-hand side

$$C(\epsilon \sup_{[0,T]} |v|_{C^2(M)} + |w|_{C^{2,\alpha}(M)} + \sup_{[0,T]} |u|_{C^\alpha(M)})$$

which implies the estimate of v

$$(5.33) \quad \sup_{[0,T]} |\partial_t v|_{C^\alpha(M)} + \sup_{[0,T]} |v|_{C^{2,\alpha}(M)} \leq C(|w|_{C^{2,\alpha}(M)} + \sup_{[0,T]} |u|_{C^\alpha(M)})$$

provided $C\epsilon < \frac{1}{2}$. \square

Proof. (proof of Theorem 5.22) Now we complete our proof of Theorem 5.22 by contraction. If the time of existence cannot be extended beyond some $t_0 < T$, then the $C^{2,\alpha}$ norm must blow up at time t_0 . This contradicts the a priori $C^{2,\alpha}$ estimates, Proposition 5.32, we derived. \square

Finally we consider the time regularity of the solution.

Lemma 5.33. *Let $w \in C^{2,\alpha}$ and $u \in C^0([0, T], C^{2,\alpha})$. Then the solution of (5.14) stays in*

$$C^0([0, T], C^{2,\alpha}) \cap C^1([0, T], C^\alpha).$$

Proof. Choose two sequences $w_n \in C^{\alpha+\epsilon}(M)$ and $u_n \in C^0([0, T], C^{\alpha+\epsilon}(M))$ for $0 \leq \epsilon < 1 - \alpha$ such that

$$w_n \rightarrow w \in C^\alpha(M) \text{ and } u_n \rightarrow u \in C^0([0, T], C^\alpha(M)).$$

According to Theorem 5.21 and the estimate (5.22), for each n there exists a solution $v_n \in C^{2+\alpha+\epsilon, 1}(M \times [0, T])$ of the linearized equation

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta_\varphi v - Q(v) = u_n \text{ in } Q_T, \\ \Delta_\varphi Q(v) = v^{i\bar{j}} T_{i\bar{j}} \text{ in } Q_T, \\ v(0) = w_n. \end{cases}$$

It is directly verified that $v_n \in C^0([0, T], C^{2,\alpha}) \cap C^1([0, T], C^\alpha)$. Moreover, (5.22) implies for any $t, s \in [0, T]$,

$$\sup_{[0,T]} |v_m - v_n|_{C^{2,\alpha}(M)} \leq C(|w_m - w_n|_{C^{2,\alpha}(M)} + \sup_{[0,T]} |u_m - u_n|_{C^\alpha(M)}).$$

From this we immediately conclude that v_n is a Cauchy sequence in $C^0([0, T], C^{2,\alpha})$. Hence we obtain $v \in C^0([0, T], C^{2,\alpha})$ and the lemma follows from (5.14). \square

5.6. Regularity of the pseudo-Calabi flow. We start with recalling an interior regularity theorem for the linear equation in [42] and [43]. Let $Q = M \times (T_1, T_2)$ and $Q' = M \times (T_3, T_4) \subset Q$, where $0 < T_1 < T_3 < T_4 < T_2$.

Theorem 5.34. (*L^p interior estimate [42][43]*) Let $u \in W_{loc}^{k+2,p}(Q)$ for some $p \geq 2$ be the solution of the linear equation (5.18) that satisfies the following conditions

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 > 0 \text{ for any } \xi \in R^n \setminus \{0\},$$

$$|a^{ij}|_{W^{k,\infty}(Q)} + |b^i|_{W^{k,\infty}(Q)} + |c|_{W^{k,\infty}(Q)} \leq M$$

and

$$|a^{ij}(X) - a^{ij}(Y)| = \omega\left(\frac{|X - Y|}{d(X, Y)}\right)$$

for some positive constants λ and M and a positive, continuous, increasing function ω with $\omega(0) = 0$. Then for any $T_3 - T_1 > \epsilon$ we have

$$|u|_{W^{k+2,p}(Q')} \leq C(k+2, n, p, \lambda, M, \epsilon)(|u|_{L^p(Q)} + |f|_{W^{k,p}(Q)}).$$

Consider the linear parabolic equation of divergence form

$$(5.34) \quad u_t = g^{p\bar{q}}u_{p\bar{q}} + b^i u_i + cu + f.$$

Here $g_{p\bar{q}}$ is a Kähler metric and g_{ij} for $1 \leq i, j \leq 2n$ is the corresponding J -invariant Riemannian metric. Similarly to Theorem 5.16, we can generalize the first part of Theorem 5.15 on a Riemannian manifold M .

Theorem 5.35. Let $f \in UC(M \times [0, t_0]) \cap C^{\alpha,0}(M \times [0, t_0])$ and $u_0 \in C^0(M)$. If the coefficients g^{ij} , b^i and c belong to $C^\alpha(M)$ with $0 < \alpha < 1$, and $g^{i\bar{j}}$ satisfies $\Lambda|\xi|^2 \geq g^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 > 0$ for any $\xi \in R^n \setminus \{0\}$. If $u_0 \in C^0(M)$, then (5.34) has a unique solution which belongs to $C([0, T] \times \Omega)$. For every $\epsilon \in (0, t_0]$ there is a constant C such that

$$||u||_{C^{2+\alpha,1}(M \times [\epsilon, t_0])} \leq C(\epsilon)(|u_0|_{C^0(M)} + |f|_{C^{\alpha,0}(M \times [0, t_0])}).$$

Denote $Q_{T_\epsilon} = M \times [\epsilon, T]$ where $0 < \epsilon < T$; we have the following result.

Proposition 5.36. The $C^{2,1}(Q_{T_\epsilon}) \cap C^0([\epsilon, T], C^2(M))$ solution of the pseudo-Calabi flow belongs to

$$C^0([\epsilon, T], C^{2+\alpha}(M)) \cap C^1([\epsilon, T], C^\alpha(M)).$$

Proof. Since we assume $\varphi \in C^2(Q_{T_\epsilon})$, applying the L^p estimate to the equation (3.6) with the normalization condition (3.7), we have for any $1 < p < \infty$

$$P \in C^0([\epsilon, T], W^{2,p}(M)).$$

Then the embedding theorem implies that P belongs to $C^0([\epsilon, T], C^{1,\alpha}(M))$.

In order to obtain a higher regularity, we consider the equation in a coordinate chart (\mathcal{U}, x_k) for $1 \leq k \leq 2n$, and select a direction $e_k = \frac{\partial}{\partial x^k}$ in the tangent space $T_R M$. We define the difference quotient of φ at x in the direction e_k as

$$\varphi_\rho = \frac{\varphi(x + \rho e_k, t) - \varphi(x, t)}{\rho}.$$

Taking the difference quotients of both sides of the equation (5.1), we obtain

$$\begin{aligned}
 (5.35) \quad \frac{\partial \varphi_\rho}{\partial t} &= \frac{h(x + \rho e_k, t) - h(x, t)}{\rho} + \frac{P(x + \rho e_k, t) - P(x, t)}{\rho} \\
 &= \frac{\log \frac{\omega_\varphi^n(x + \rho e_k, t)}{\omega_\varphi^n(x, t)} - \log \frac{\omega^n(x + \rho e_k, t)}{\omega^n(x, t)}}{\rho} + \frac{P(x + \rho e_k, t) - P(x, t)}{\rho} \\
 &= \frac{1}{\rho} \log \frac{\omega_\varphi^n(x + \rho e_k, t)}{\omega_\varphi^n(x, t)} - \frac{1}{\rho} \log \frac{\omega^n(x + \rho e_k, t)}{\omega^n(x, t)} + \partial_k P(x + \vartheta e_k, t).
 \end{aligned}$$

We can see that the second term is smooth and the third term belongs to $C^0(Q_{T_\epsilon})$.

Now we deal with the first term. In the local coordinate chart (\mathcal{U}, x_k) for $1 \leq k \leq 2n$, we can choose some smooth real valued function ψ such that $\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi$. Then the first term can be expressed as

$$\begin{aligned}
 \frac{1}{\rho} \log \frac{\omega_\varphi^n(x + \rho e_k, t)}{\omega_\varphi^n(x, t)} &= \frac{1}{\rho} \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})(x + \rho e_k, t)}{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})(x, t)} \\
 &= \frac{1}{\rho} \int_0^1 \frac{\partial}{\partial \theta} \log \det(g_{\theta i\bar{j}}) d\theta \\
 &= \int_0^1 g_{\varphi\theta}^{i\bar{j}} d\theta (\varphi_\rho + \psi_\rho)_{i\bar{j}},
 \end{aligned}$$

where $g_{\varphi\theta} = \theta g_\varphi(x + \rho e_k, t) + (1 - \theta) g_\varphi(x, t)$ and $\psi_\rho = \frac{\psi(x + \rho e_k, t) - \psi(x, t)}{\rho}$. Since

$$a^{ij} = \int_0^1 g_{\varphi\theta}^{i\bar{j}} d\theta \in C^0(Q_{T_\epsilon})$$

is uniformly elliptic and $\partial_k P \in C^0(Q_{T_\epsilon})$, Proposition 5.34 implies

$$\varphi_\rho \in W^{2,p}(Q_{T_\epsilon})$$

for any $p > 1$. After letting $\rho \rightarrow 0$, we get $\frac{\partial \varphi}{\partial x^k}$ in $W^{2,p}(Q_{T_\epsilon})$. The imbedding theorem further implies $\frac{\partial \varphi}{\partial x^k} \in C^{1,\alpha}(Q_{T_\epsilon})$. So we have

$$\varphi \in C^0([\epsilon, T], C^{2+\alpha}(M)).$$

Since $P \in C^0([\epsilon, T], C^{1,\alpha}(M))$, by using the equation (5.1) we have

$$\partial_t \varphi \in C^0([\epsilon, T], C^\alpha(M)).$$

Hence the proposition follows. \square

Proposition 5.37. *The $C^0([\epsilon, T], C^{2+\alpha}(M)) \cap C^1([\epsilon, T], C^\alpha(M))$ solution of the pseudo-Calabi flow equation belongs to $C^\infty(M \times [2\epsilon, T])$.*

Proof. Lemma 5.6 implies $P \in C^0([\epsilon, T], C^{2+\alpha}(M))$. Then we have

$$\partial_k P \in C^0([\epsilon, T], C^\alpha(M)) \text{ and } a^{ij} \in C^0([\epsilon, T], C^\alpha(M)).$$

We consider the equation by freezing the coefficient of (5.35),

$$\begin{aligned}
 (5.36) \quad \frac{\partial \varphi_\rho}{\partial t} &= \triangle \varphi_\rho + \left(\int_0^1 g_{\varphi\theta}^{i\bar{j}} d\theta - g^{i\bar{j}} \right) \varphi_{\rho i\bar{j}} + \int_0^1 g_{\varphi\theta}^{i\bar{j}} d\theta \psi_{\rho i\bar{j}} \\
 &\quad - \frac{1}{\rho} \log \frac{\omega^n(x + \rho e_k, t)}{\omega^n(x, t)} + \partial_k P(x + \vartheta e_k, t).
 \end{aligned}$$

Therefore, we obtain $\partial_k \varphi \in C^0([\epsilon, T], C^{2,\alpha}(M))$ by Theorem 5.36 and the condition (5.12) in each coordinate chart (\mathcal{U}, x_k) . Moreover, by using (5.35) we deduce that

$$\partial_t \varphi \in C^0([\epsilon, T], C^{1,\alpha}(M)).$$

Repeating the same process again and again, we have

$$\varphi \in C^0([\epsilon \sum_{p=0}^q \frac{1}{2^p}, T], C^{q+2,\alpha}(M))$$

for any $q = 1, 2, \dots$. Then taking the derivative of (5.1) with respect to variable t , we have $\varphi \in C^2([2\epsilon, T], C^\infty(M))$. By iteration of this procedure, the proposition follows. \square

Since ϵ can be arbitrarily small, we have the following theorem.

Theorem 5.38. *The $C^0([0, T], C^{2+\alpha}(M)) \cap C^1([0, T], C^\alpha(M))$ solution of the pseudo-Calabi flow equation in fact belongs to $C^\infty(M \times (0, T])$.*

Remark 5.6. The proof still holds if one uses the estimate (5.24) instead of Theorem 5.15.

6. LONG TIME EXISTENCE OF THE PSEUDO-CALABI FLOW

In this section we shall use C_i for $i = 1, 2, \dots$ to distinguish different generic constants.

Theorem 6.1. *Let T be a finite time. If $g_\varphi(t)$ is a solution of the pseudo-Calabi flow with $\text{Ric}_\varphi(t)$ uniformly bounded for all time $t \in [0, T]$, then the pseudo-Calabi flow can be extended past the time T .*

Proof. Because of the short time existence theorem, we know that the pseudo-Calabi flow can be restarted at time T if the solution then belongs to $C^{2,\alpha}(M)$. Theorem 6.1 will thus follow from the a priori estimates that we shall present in Proposition 6.8. \square

Lemma 6.2. *Suppose that along the pseudo-Calabi flow, there holds $\int_0^t S_\varphi(t) dt \geq -C_1$ for some constant C_1 . Then there exists a constant C_2 depending on C_1, T and $\sup_M \log \frac{\omega_{\varphi_0}^n}{\omega^n}$ such that*

$$\sup_M h \leq C_2.$$

Proof. Since the t integral of the scalar curvature is uniformly bounded below, and

$$\partial_t h = \Delta_\varphi \partial_t \varphi = -S_\varphi + \underline{S},$$

then $\sup_M h \leq C_1 + \sup_M h(0)$. \square

Denote the average of φ with respect to ω by $\underline{\varphi} = \int_M \varphi \omega^n$.

Lemma 6.3. *There exists a constant C_3 such that*

$$\sup_M (\varphi - \underline{\varphi}) \leq C_3.$$

Proof. Since $\Delta\varphi + n > 0, \forall t \geq 0$, using the Green representation we have

$$\begin{aligned}\varphi(x) - \underline{\varphi} &= -\frac{1}{V} \int_M \Delta\varphi(y) G(x, y) \omega^n(y) \\ &\leq n \frac{1}{V} \int_M G(x, y) \omega^n(y).\end{aligned}$$

Since $0 \leq G(x, y) \leq \frac{C_4}{d(x, y)^{2n-2}}$, we have

$$\sup_{M \times [0, T]} (\varphi - \underline{\varphi}) \leq C_3 = n \frac{1}{V} \int_M G(x, y) \omega^n(y).$$

□

The lower bound of the normalized potential is obtained by Yau's C^0 estimate.

Theorem 6.4. (Yau [55]) *The lower bound of $\varphi - \underline{\varphi}$ is controlled by the upper bound of h ,*

$$(6.1) \quad \inf_M (\varphi - \underline{\varphi}) \geq -C_5 e^{C_6 \sup_M h}.$$

Here C_5 and C_6 depend only on ω and $\sup_M (\varphi - \underline{\varphi})$.

Theorem 6.5. (Chen-Tian [20]) *Suppose Ric_φ is bounded from below, then*

$$(6.2) \quad \inf_M h \geq -4C_7 e^{2(1 + \int_M h \omega_\varphi^n)}.$$

Combining Lemma 6.2 and (6.2), we get the uniform bound of h :

$$(6.3) \quad \sup_M |h| \leq C_2 + 4C_7 e^{2(1 + C_2 V)}.$$

We deduce the lower bound of the average of φ by the normalization condition.

Lemma 6.6. *The following estimate holds*

$$\inf_{[0, T]} \underline{\varphi} \geq \underline{\varphi}(0) + T \inf_M h.$$

Proof. Due to the normalization condition $\int_M e^P \omega^n = \text{Vol}$, we apply the Jensen's inequality to get

$$\int_M P \omega^n \leq 0.$$

So we obtain

$$\frac{\partial}{\partial t} \underline{\varphi} = \frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} \omega^n = \frac{1}{V} \int_M (h - P) \omega^n \geq \inf_M h.$$

Similarly, we get $\int_M h \omega^n \leq 0$ since $\int_M e^h \omega^n = V$. Therefore the lemma follows from $\inf_M h \leq 0$. □

Plugging the above lemma and Lemma 6.3 in (6.1), we get

$$\inf_M \varphi \geq -C_5 e^{C_6 \sup_M h} + \underline{\varphi}(0) + T \inf_M h.$$

Then by (6.3) we have that φ is uniformly bounded below. On the other hand (6.1) further implies that for a fixed point $p \in M$

$$\underline{\varphi} \leq \inf_M \varphi + C_5 e^{C_6 \sup_M h} \leq \varphi(p) + C_5 e^{C_6 \sup_M h}.$$

So the upper bound of φ follows from Lemma 6.3, i.e.

$$\sup_M \varphi \leq \underline{\varphi} + C_3.$$

Theorem 6.7. (*Chen-He [18]*) *Suppose that the Ricci curvature Ric_φ is uniformly bounded above. Then there exist two constants C_8 and C_9 such that*

$$n + \Delta\varphi \leq C_8 e^{C_9 \cdot \text{Osc}_M \varphi + \frac{\sup_M h}{n-1}}.$$

Working with normal coordinates, we have

$$\frac{1}{1 + \varphi_{i\bar{i}}} = \frac{\prod_{j \neq i} (1 + \varphi_{j\bar{j}})}{\prod_i (1 + \varphi_{i\bar{i}})} \leq \left(\frac{n + \Delta\varphi}{n-1} \right)^{n-1} e^{-h} \leq C_{10} e^{C_{11} \text{Osc}_M \varphi + \text{Osc}_M h}.$$

So the metrics are all equivalent, that is

$$C_{10} e^{-C_{11} \text{Osc}_M \varphi - \text{Osc}_M h} g_{i\bar{j}} \leq g_{\varphi i\bar{j}} \leq C_8 e^{C_9 \text{Osc}_M \varphi + \frac{\sup_M h}{n-1}} g_{i\bar{j}}.$$

By the Ricci curvature bounds

$$-C_1 g_{\varphi i\bar{j}} \leq R_{\varphi i\bar{j}} \leq C_2 g_{\varphi i\bar{j}},$$

we have that Δh is bounded; namely

$$-C_2(n + \Delta\varphi) + S \leq \Delta h = -g^{i\bar{j}} R_{\varphi i\bar{j}} + S \leq C_1(n + \Delta\varphi) + S.$$

This together with the fact that $\sup_{M \times [0, T]} |h| \leq C$ implies that $h \in W^{2,p}(M)$ for any $p > 1$ and $t \in [0, T]$. Then the Evans-Krylov estimate [33][41] shows that φ has uniform $C^{2,\alpha}$ bound. The argument we made so far is summarized in the statement here below.

Proposition 6.8. *Let T be a finite time. If $g_\varphi(t)$ is a solution of the pseudo-Calabi flow with $Ric_\varphi(t)$ uniformly bounded for all time $t \in [0, T]$, then there exists a constant C depending on T such that $\sup_{[0, T]} |\varphi(t)|_{C^{2,\alpha}} \leq C$ for some $0 < \alpha < 1$.*

7. PSEUDO-CALABI FLOW IN THE SPACE OF KÄHLER METRICS

Suppose that $\varphi(t)$ for $0 \leq t < T$ is the solution of (3.4) given by Theorem 5.1 and T is the maximal existence time. In this section, we consider the following system of equations,

$$(7.1) \quad \begin{cases} \frac{\partial}{\partial t} \psi = \log \frac{\omega_\psi^n}{\omega^n} - P(\psi) - \bar{h} + \bar{P}, \\ \Delta_\psi P(\psi) = \text{tr}_\psi Ric(\omega) - \underline{S}, \\ \psi(0) = \varphi_0, \end{cases}$$

with the normalization condition

$$\int_M e^P \omega^n = \int_M e^{\frac{\partial \psi}{\partial t} + P + \bar{h} - \bar{P}} \omega^n = \text{Vol}(M).$$

Here $\bar{h} = \frac{1}{V} \int_M \log \frac{\omega_\psi^n}{\omega^n} \omega_\psi^n$ and $\bar{P} = \frac{1}{V} \int_M P(\psi) \omega_\psi^n$. Actually (7.1) is obtained by replacing φ with

$$\psi = \varphi + \int_0^t (-\bar{h} + \bar{P}) ds$$

in (3.5). Since $\partial_t I(\psi) = \int_M \partial_t \psi \omega_\varphi^n = 0$ (see (3.1)), if we further assume $I(\varphi_0) = 0$ then ψ always stays in \mathcal{H}_0 .

We assume that M admits a cscK metric ω . We choose ω as the reference metric. Then we shall show that if φ_0 is in a sufficiently small neighborhood of the zero function, then $\varphi(t)$ can be extended and ω_φ always stays in a small neighborhood of ω . Before we go into the details of the proof, we cite the theorems we will use later.

Recall the explicit form of K -energy in Chen [10] and Tian [52]

$$\begin{aligned} \nu_\omega(\varphi) &= \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n + \frac{S}{V} \sum_{i=0}^n \frac{n!}{(i+1)!(n-i)!} \int_M \varphi \omega^{n-i} \wedge (\partial\bar{\partial}\varphi)^i \\ &\quad - \frac{1}{V} \sum_{i=0}^{n-1} \frac{n!}{(i+1)!(n-i-1)!} \int_M \varphi Ric \wedge \omega^{n-1-i} \wedge (\partial\bar{\partial}\varphi)^i. \end{aligned}$$

So the K -energy is well defined for any L^∞ Kähler metrics. Its derivative is

$$\nu'_\omega(v) = \int_M \log \frac{\omega_\varphi^n}{\omega^n} \Delta_\varphi v \omega_\varphi^n - \frac{1}{(n-1)!} \int_M v Ric(\omega_0) \wedge \omega_\varphi^{n-1} + \frac{S}{V} \int_M v \omega_\varphi^n.$$

Here v is the infinitesimal variation of φ .

Chen-Tian further proved that

Theorem 7.1. (*Chen-Tian [22]*) *Let M be a compact Kähler manifold with a cscK metric ω . Then $\nu_\omega(\varphi) \geq 0$ for any φ with $\omega_\varphi > 0$.*

They also proved the uniqueness of the extremal metrics.

Theorem 7.2. (*Chen-Tian [22]*) *Let $(M, [\omega])$ be a compact Kähler manifold with a Kähler class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. Then there is at most one extremal Kähler metric with Kähler class $[\omega]$ modulo holomorphic transformations. Namely, if ω_1 and ω_2 are two extremal Kähler metrics with the same Kähler class, then there is a holomorphic transformation σ such that $\omega_1 = \sigma^* \omega_2$.*

7.1. M admits no holomorphic vector field. Suppose that φ is the solution of (7.1) and T is the time when

$$(7.2) \quad |\varphi|_{C^{2,\alpha}} \leq \epsilon_1 \text{ for all } t \in [0, T].$$

Letting t_0 be a small time, we apply Theorem 5.38 to obtain the higher order uniform bound of φ ; namely

$$|\psi_s|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1, g, t_0) \text{ for all } t \in [T - t_0, T + t_0].$$

Then we introduce the space

$$\mathcal{S} = \{\varphi \mid |\varphi|_{C^{2,\alpha}} \leq \epsilon_1; |\varphi|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1, g, t_0)\}.$$

Clearly, $0 \in \mathcal{S}$. In this subsection we are going to prove the following theorem.

Theorem 7.3. *Assume M admits a cscK metric ω and has no holomorphic vector fields. For any $\epsilon_1 > 0$, there exists $\epsilon_0 > 0$ such that if $|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0$, the lifespan of the solution is $T = \infty$ and we have that $|\psi(t)|_{2,\alpha} < \epsilon_1$ for all $t \in [0, +\infty)$.*

Proof. Suppose that the conclusion fails, then there must exist a sequence of initial data φ_s^0 such that

$$|\varphi_s^0|_{C^{2,\alpha}} \leq \frac{1}{s}.$$

By virtue of the short time existence Theorem 5.1, we get a sequence of solutions $\psi_s(t)$ satisfying the equations (7.1) with $\psi_s(0) = \varphi_s^0$. Let T_s be the first time that

$$(7.3) \quad |\psi_s(T_s)|_{C^{2,\alpha}} = \epsilon_1 \text{ and } |\psi_s(t)|_{C^{2,\alpha}} < \epsilon_1 \text{ on } [0, T_s).$$

According to Theorem 5.3, we have that $\inf_s T_s > 0$ uniformly and there is a uniformly small time t_0 such that

$$|\psi_s|_{C^{2,\alpha}(M)} \leq C(\epsilon_1, t_0), \forall t \in [T_s - t_0, T_s + t_0].$$

Moreover, from the regularity Theorem 5.38, we obtain the higher order uniform bound of the sequence of the solutions

$$(7.4) \quad |\psi_s|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1, g, t_0), \forall t \in [T_s - \frac{t_0}{2}, T_s + \frac{t_0}{2}].$$

Therefore we can choose a subsequence of $\phi_s = \psi_s(T_s)$; we use the same index for convenience, so that

$$\phi_s \rightarrow \phi_\infty \text{ in } C^{k,\alpha}, \forall k \geq 0.$$

Since (7.3) still holds in the limit, we have

$$(7.5) \quad |\phi_\infty|_{C^{2,\alpha}} = \epsilon_1.$$

Note that the K -energy is well defined along $\psi_s(t)$. By using Theorem 7.1 and the decrease of the K -energy along the flow, we conclude that

$$0 \leq \nu_\omega(\phi_s) \leq \nu_\omega(\psi_s(0)) \leq \frac{C}{s}$$

which implies

$$\lim_{s \rightarrow \infty} \nu_\omega(\psi_s) = \nu_\omega(\psi_\infty) = 0.$$

So Theorem 7.2 implies that $\phi_\infty = \text{const}$. Furthermore from $I(\phi_\infty) = 0$, we deduce that $\phi_\infty = 0$. This contradicts (7.5). \square

Since $|\psi(t)|_{C^{2,\alpha}} \leq \epsilon_1$ is uniformly bounded, we have that $|\psi(t)|_{C^k} \leq C_k$ for any $k \geq 3$ away from $t = 0$. Similarly to the argument above, for any sequence t_i we can extract a subsequence (still denoted by ψ_{t_i}) such that $\psi(t_j)$ converge to a limit function ψ_∞ in C^∞ norm. Also the limiting metric ω_{ψ_∞} is a cscK metric. Since we assume that M admits no holomorphic vector field, according to Theorem 7.2 we deduce that $\psi_\infty = 0$ from the normalization condition $I(\psi_\infty) = 0$. Therefore the pseudo-Calabi flow converges to the original cscK metric, since t_i is chosen randomly. In conclusion, the picture in the space of Kähler metric \mathcal{H}_0 is that the pseudo-Calabi flow will shrink to the unique cscK metric if the initial potential is close around it.

7.2. M admits holomorphic vector fields. When M admits holomorphic vector fields, the contradiction argument is much more sophisticated. We denote the subset of \mathcal{H}_0 that contains the potentials of all cscK metrics by

$$\mathcal{E}_0 = \{\rho \in C^\infty(M, R) | \sigma^* \omega = \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \rho \text{ and } I(\varphi) = 0 \text{ for any } \sigma \in \text{Aut}_0(M)\}.$$

Mabuchi [46] proved that it is a finite-dimensional totally geodesic submanifold in \mathcal{H}_0 . Then for each $\varphi \in \mathcal{H}_0$, there exists a unique ρ that minimizes the distance from φ to \mathcal{H}_0 , i.e.

$$\text{dist}(\varphi, \rho) = \text{dist}(\varphi, \mathcal{E}_0).$$

Meanwhile there is a $\sigma \in \text{Aut}_0(M)$ such that $\sigma^*\omega_\varphi = \omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}(\varphi(\sigma) + \rho) \in [\omega]$. We assume $\psi(0)$ stays in the complement of \mathcal{H}_0 ; otherwise the flow keeps fixed. Furthermore we can assume that $\rho_0 = 0$; if not, we can replace the background metric ω with ω_{ρ_0} .

Lemma 7.4. *There exists a small constant ϵ and a positive constant C_2 such that if ρ satisfies $d(0, \rho) \leq \epsilon$, then $|\rho|_{C^{3,\alpha}} \leq C_2\epsilon$.*

Proof. In the Riemannian manifold \mathcal{E}_0 , any small ϵ neighborhood near $\rho = 0$ can be pulled back by the exponential map \exp_0 to the tangent space $T_0(\mathcal{E}_0)$ near 0. Denote $\psi = \exp_0^{-1}(\rho)$. Note that all norms on a finite-dimensional vector space are equivalent, so the norm induced by the distance on $T_0(\mathcal{E}_0)$ is equivalent to the $C^{3,\alpha}$ norm (here the point in $T_0(\mathcal{E}_0)$ is also a function on M). So $d(0, \rho) \leq \epsilon$ implies that $|\exp_0^{-1}(\rho)|_{C^{3,\alpha}}$ is bounded by $C_1\epsilon$. Furthermore, let ϵ be small enough such that ψ is a diffeomorphism in the ϵ neighborhood near $\rho = 0$; then there exists a constant C_2 such that $|\rho|_{C^{3,\alpha}} \leq C_2\epsilon$. Hence this lemma follows. \square

Remark 7.1. In fact, we can improve the above conclusions in Lemma 7.4 for C^k of any fix $k \geq 0$.

Lemma 7.5. *Suppose that $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$ for some small constant ϵ_1 depending on ϵ . Then there exists a constant C depending on ϵ_1 such that $|\rho|_{C^{3,\alpha}} \leq C$ and $|\sigma|_h \leq C$. Here h is the left invariant metric in $\text{Aut}(M)$.*

Proof. we define a path by $\gamma_t = t\varphi - I(t\varphi) \in \mathcal{E}_0$ for $0 \leq t \leq 1$. It is obvious that this path stays in \mathcal{H}_0 . Then since $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$, we have

$$\begin{aligned} d(0, \varphi) &\leq L(\gamma_t) = \int_0^1 \left(\int_M \left(\frac{\partial \gamma_t}{\partial t} \right)^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \\ &= \int_0^1 \left(\int_M (\varphi - \partial_t I(t\varphi))^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \leq C_3\epsilon_1. \end{aligned}$$

Moreover, since ρ realizes the shortest distance from φ to \mathcal{H}_0 , by using the triangle inequality we obtain

$$\begin{aligned} d(0, \rho) &\leq d(0, \varphi) + d(\varphi, \rho) \\ (7.6) \quad &\leq 2d(0, \varphi) \leq C_3\epsilon_1. \end{aligned}$$

Applying Lemma 7.4 with $\epsilon = C_3\epsilon_1$ we have $|\rho|_{C^{3,\alpha}} \leq C_2C_3\epsilon_1$. Furthermore, from Lemma 4.6 in Chen-Tian [21], we obtain $|\sigma|_h$ is also bounded. Here h is the left invariant metric in $\text{Aut}(M)$. Therefore the lemma holds for some constant C . \square

Now we prove the invariance of the K -energy.

Lemma 7.6. $\nu(\omega, \omega_{(\sigma^{-1})^*(\varphi-\rho)}) = \nu(\omega, \omega_\varphi)$.

Proof. Since the K -energy is invariant under the holomorphic transformation, we get

$$\nu(\omega, \omega_{(\sigma^{-1})^*(\varphi-\rho)}) = \nu(\sigma^*\omega, \omega_\varphi) = \nu(\omega_\rho, \omega_\varphi).$$

Then the 1-cocycle condition of the K -energy (Theorem (2.4) in [45]) gives

$$\nu(\sigma^*\omega, \omega_\varphi) = \nu(\omega_\rho, \omega) + \nu(\omega, \omega_\varphi).$$

Since both ω and ω_ρ are cscK metrics, the lemma follows from $\nu(\omega_\rho, \omega) = 0$. \square

Lemma 7.7. *For any $\epsilon > 0$, There exists a small constant δ such that for any $\varphi \in \mathcal{S}$, if $\nu_\omega(\varphi) \leq \delta$, then $|(\sigma^{-1})^*(\varphi - \rho)|_{C^{2,\alpha}} < \epsilon$.*

Proof. If the conclusion fails, we assume there is a sequence φ_s with

$$|\varphi_s|_{C^{2,\alpha}(M,g)} \leq \epsilon_1, |\varphi_s|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1, g, t_0), \text{ and } \nu_\omega(\varphi_s) \leq \frac{1}{s}$$

such that

$$(7.7) \quad |(\sigma_s^{-1})^*(\varphi_s - \rho_s)|_{C^{2,\alpha}} \geq \epsilon_1.$$

We denote $\hat{\varphi}_s = (\sigma_s^{-1})^*(\varphi_s - \rho_s)$. After making use of Lemma 7.5 and Lemma 7.6, we can choose some subsequence of $\hat{\varphi}_s$ such that

$$\hat{\varphi}_s \rightarrow \hat{\varphi}_\infty \in C^l \text{ for any } l \geq 0 \text{ and } \nu_\omega(\hat{\varphi}_\infty) = 0.$$

Therefore we have $\hat{\varphi}_\infty \in \mathcal{E}_0$ by Chen-Tian [22]. And then $\varphi_\infty \in \mathcal{E}_0$.

We claim that $d(\varphi_\infty, \rho_\infty) = 0$. Otherwise there is some sufficient large N such that, for any $s > N$, the sequence $d(\varphi_s, \rho_s) = d(\varphi_s, \mathcal{E}_0)$ has positive lower bound. From the fact that the distance function is at least C^1 we have $d(\varphi_\infty, \mathcal{E}_0) > 0$, that contradicts $\varphi_\infty \in \mathcal{E}_0$.

This claim implies $\hat{\varphi}_\infty = 0$. But (7.7) gives $|\hat{\varphi}_\infty|_{C^{2,\alpha}} \geq \epsilon_1 > 0$. It is a contradiction. \square

Remark 7.2. In fact, we can improve the above conclusion to get

$$|(\sigma^{-1})^*(\varphi - \rho)|_{C^{k,\alpha}} < \epsilon$$

for any $k \geq 0$. Then combining with Theorem 8.1, we can show that the solution will stay in a small neighborhood and exponentially decay to a unique cscK metric by directly obtaining the estimate of the solution.

Remark 7.3. We use the theorem of Chen-Tian [22] that the cscK metric is the global minimizer of the K -energy here. Actually, we only need a local version. I.e. the cscK metric is the local minimizer of the K -energy. That can be proved by using the non-negativeness of the hessian of the K -energy and the geometry of the critical submanifold.

Theorem 7.8. *Assume that M admits a cscK metric ω and has nontrivial holomorphic vector fields. If $|\psi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0$, then there exists a holomorphic transformation $\varrho(t)$ such that $\varrho(t)^*\omega_{\psi_t}$ stays in a small neighborhood of ω . Moreover, for any sequence $g_{\psi_{t_i}}$, one can extract a subsequence $g_{\psi_{t_{i_j}}}$ such that $\varrho_j^*g_{\psi_{t_{i_j}}}$ converges to a cscK metric.*

Proof. According to the short time existence Theorem 5.1, we assume that T is the first time when the following holds

$$|\psi|_{C^{2,\alpha}} < \epsilon_1 \text{ on } [0, T) \text{ and } |\psi(T)|_{C^{2,\alpha}} = \epsilon_1.$$

Now there are two cases. If $\psi(T)$ is a cscK metric, then the flow will stop right at T and our theorem is proved. Otherwise, we will extend the flow as follows. By virtue of Theorem 5.38, we obtain the higher order uniform bound of the solutions

$$|\psi|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1, g, t_0), \forall t \in [T - \frac{t_0}{2}, T + \frac{t_0}{2}].$$

So we have $\psi(T) \in \mathcal{S}$.

Now we firstly choose ϵ_0 small enough to guarantee

$$\nu_\omega(\psi_0) \leq o.$$

Since the K -energy is decreasing along the pseudo-Calabi flow, we get so

$$\nu_\omega(\psi(T)) \leq \nu_\omega(\psi_0).$$

Let σ be the projection of $\psi(T)$ onto \mathcal{H}_0 . Then due to Lemma 7.7 we obtain

$$(7.8) \quad |(\sigma^{-1})^*(\psi(T) - \rho_1)|_{C^{2,\alpha}} < \epsilon_0.$$

Next we show the equation is invariant under the holomorphic transformation. By (7.1), $\psi(t) - \rho_1$ is the solution of

$$(7.9) \quad \begin{cases} \frac{\partial}{\partial t} \psi = \log \frac{\omega_{\psi+\rho_1}^n}{\omega^n} - P(\psi + \rho_1) - \overline{\log \frac{\omega_{\psi+\rho_1}^n}{\omega^n}} + \overline{P(\psi + \rho_1)}, \\ \Delta_{\psi+\rho_1} P(\psi + \rho_1) = tr_{\psi+\rho_1} Ric(\omega) - \underline{S}. \end{cases}$$

Note that ω_{ρ_1} is a cscK metric satisfying

$$(7.10) \quad \begin{cases} \log \frac{\omega_{\rho_1}^n}{\omega^n} = P(\rho_1), \\ \Delta_{\rho_1} P(\rho_1) = tr_{\rho_1} Ric(\omega) - \underline{S}. \end{cases}$$

Letting $P_1(\psi + \rho_1) = P(\psi + \rho_1) - P(\rho_1)$, we have

$$\begin{aligned} \Delta_{\psi+\rho_1} P_1(\psi + \rho_1) &= \Delta_{\psi+\rho_1} (P(\psi + \rho_1) - P(\rho_1)) \\ &= tr_{\psi+\rho_1} Ric(\omega) - \underline{S} - \Delta_{\psi+\rho_1} P(\rho_1) \\ &= tr_{\psi+\rho_1} Ric(\omega_{\rho_1}) - \underline{S}. \end{aligned}$$

From this we obtain new equations from $t = T$ by combining (7.9) and (7.10) so that $\psi(t) - \rho_1$ satisfies

$$(7.11) \quad \begin{cases} \frac{\partial}{\partial t} \psi = \log \frac{\omega_{\psi+\rho_1}^n}{\omega_{\rho_1}^n} - P_1(\psi + \rho_1) - \overline{\log \frac{\omega_{\psi+\rho_1}^n}{\omega_{\rho_1}^n}} + \overline{P_1(\psi + \rho_1)}, \\ \Delta_{\psi+\rho_1} P_1(\psi + \rho_1) = tr_{\psi+\rho_1} Ric(\omega_{\rho_1}) - \underline{S}. \end{cases}$$

After taking transformation $(\sigma^{-1})^*$ of (7.11) we have $\psi_1 = (\sigma^{-1})^*(\psi(t) - \rho_1)$ is the solution of

$$(7.12) \quad \begin{cases} \frac{\partial}{\partial t} \psi_1 = \log \frac{\omega_{\psi_1}^n}{\omega^n} - P_1(\psi_1) - \overline{\log \frac{\omega_{\psi_1}^n}{\omega^n}} + \overline{P_1(\psi_1)}, \\ \Delta_{\psi_1} P_1(\psi_1) = tr_{\psi_1} Ric(\omega) - \underline{S}, \\ \psi_1(0) = \psi(T)(\sigma) - \rho_1, \end{cases}$$

with (7.8) and

$$\nu_\omega(\psi_1(0)) \leq o.$$

All averages in these equations are taken over M with respect to the metric g_{ψ_1} in these equations. Theorem 5.1 implies that (7.12) can be extended beyond T . Finally since $\psi_1(T_1) \in \mathcal{S}$, we can repeat the same steps as before by induction till ψ_s becomes a cscK metric at time T if $T < \infty$. If not, we deduce that the pseudo-Calabi flow has long time existence and there is a sequence

$$\psi_s(0) = \psi_s^0 = (\sigma_{s-1}^{-1})^*(\psi_{s-1}(T_{s-1}) - \rho_{s-1})$$

such that

$$|\psi_s^0|_{C^k} \leq C(k, \epsilon_1) \forall l \geq 0,$$

$$\lim_{s \rightarrow \infty} \nu_\omega(\psi_s^0) = \lim_{s \rightarrow \infty} \nu(\omega, \omega_{\psi_{s-1}(T_{s-1})}) = 0.$$

We further define

$$\omega_{\psi(t)} = \prod_{i=0}^{s-1} \sigma_i^* \omega_{\psi_s(t)} \text{ on } [\sum_{i=0}^{s-1} T_i, \sum_{i=0}^s T_i).$$

Then we obtain a solution $\psi(t)$ for all $t \geq 0$. In general, for any sequence $\{\psi_{t_j}\}$, there is s such that $\sum_{i=0}^{s-1} T_i \leq t_j \leq \sum_{i=0}^s T_i$. Furthermore, writing $\varrho_j = (\prod_{i=0}^{s-1} \sigma_i)^{-1}$, we have

$$|\varrho_j^* \omega_{\psi_{t_j}} - \omega|_{C^\alpha} \leq \epsilon_1 \text{ and } |\varrho_j^* \omega_{\psi_{t_j}} - \omega|_{C^k} \leq C(k, \epsilon_1).$$

Therefore all metrics are equivalent and their derivatives are bounded. It follows that there is a subsequence (with the same notation) of $\varrho_j^* g_{\psi_{t_j}}$ that converges to a limit Kähler metric g_∞ which may depend on the choice of the sequence. Since the K -energy is bounded below, we obtain

$$\lim_{s \rightarrow \infty} \nu(\omega, \omega_{\psi_{t_j}}) = 0.$$

It follows that g_∞ is a cscK metric. \square

Remark 7.4. Following the same argument in Chen-Tian [21], we can extend the holomorphic transformation ϱ to each t so that it is Lipschitz continuous in t .

8. EXPONENTIAL DECAY OF THE PSEUDO-CALABI FLOW

Recall that $\psi(t)$ is the solution of (7.1) and $g_\phi = \varrho^* g_\psi$ is the modified solution defined in Theorem 7.8. We have already proved that g_ϕ always stays in small neighborhood of ω and converges to a cscK metric sequently.

Definition 8.1. We call f -tensor the $(1, 1)$ form locally given by $f = [P_{i\bar{j}} + R_{i\bar{j}} - R_{i\bar{j}}(\omega)] dz^i \wedge d\bar{z}^j = f_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

Since $\lim_{t_j \rightarrow \infty} S_{t_j} - \underline{S} = 0$ for arbitrary subsequence $\{t_j\}$, we have

$$\lim_{t \rightarrow \infty} S_t - \underline{S} = 0.$$

Moreover, the uniform bound of $|Rm(g_\phi)|_{g_\phi}$ gives rise to the uniform bound of $|Rm(g_\psi)|_{g_\psi}$. These facts are important to prove the exponential decay of the energy functionals.

Lemma 8.1. *The following formula holds*

$$(8.1) \quad \frac{d}{dt} \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_\psi^n = -2 \frac{1}{V} \int_M (L_t \dot{\psi}, \dot{\psi}) \omega_\psi^n$$

$$+ \frac{1}{V} \int_M (-P^{i\bar{j}} - R^{i\bar{j}} + R^{i\bar{j}}(\omega)) \dot{\psi}_i \dot{\psi}_{\bar{j}} \omega_\psi^n - \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 (S - \underline{S}) \omega_\psi^n.$$

Proof. Differentiating (7.1) with respect to t we have

$$(8.2) \quad \ddot{\psi} = \Delta_\psi \dot{\psi} - \dot{P} - \underline{\Delta_\psi \dot{\psi}} - \underline{\dot{P}}.$$

Then we get

$$\begin{aligned} \frac{d}{dt} |\nabla \dot{\psi}|^2 &= f^{i\bar{j}} \dot{\psi}_i \dot{\psi}_{\bar{j}} + g_{\psi}^{i\bar{j}} \ddot{\psi}_i \dot{\psi}_{\bar{j}} + g_{\psi}^{i\bar{j}} \dot{\psi}_i \ddot{\psi}_{\bar{j}} \\ &= f^{i\bar{j}} \dot{\psi}_i \dot{\psi}_{\bar{j}} + g_{\psi}^{i\bar{j}} \Delta_{\psi} \dot{\psi}_i \dot{\psi}_{\bar{j}} - g_{\psi}^{i\bar{j}} \dot{P}_i \dot{\psi}_{\bar{j}} + g_{\psi}^{i\bar{j}} \dot{\psi}_i \Delta_{\psi} \dot{\psi}_{\bar{j}} - g_{\psi}^{i\bar{j}} \dot{\psi}_i \dot{P}_{\bar{j}}. \end{aligned}$$

On the other hand since

$$(\Delta_{\psi} \psi)_i = \psi_{k\bar{k}i} = \psi_{\bar{k}ki} = \psi_{\bar{k}ik} = \psi_{i\bar{k}k} = \Delta_{\psi} \psi_i,$$

we obtain the Laplacian of $|\nabla \dot{\psi}|^2$ given by

$$\Delta_{\psi} |\nabla \dot{\psi}|^2 = R^{i\bar{j}} \dot{\psi}_i \dot{\psi}_{\bar{j}} + g_{\psi}^{i\bar{j}} \Delta_{\psi} \dot{\psi}_i \dot{\psi}_{\bar{j}} + g_{\psi}^{i\bar{j}} \dot{\psi}_i \Delta_{\psi} \dot{\psi}_{\bar{j}} + |\dot{\psi}_{i\bar{j}}|^2 + |\dot{\psi}_{i\bar{j}}|^2.$$

In the following we use the upper index to represent the raise of the index by means of the metric g_{ψ} . Then combining these two identities we have the following evolution equation,

$$\begin{aligned} &\frac{d}{dt} \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_{\psi}^n \\ &= \frac{1}{V} \int_M [(f^{i\bar{j}} - R^{i\bar{j}}) \dot{\psi}_i \dot{\psi}_{\bar{j}} - |\dot{\psi}_{i\bar{j}}|^2 - |\dot{\psi}_{i\bar{j}}|^2 - g_{\psi}^{i\bar{j}} \dot{P}_i \dot{\psi}_{\bar{j}} - g_{\psi}^{i\bar{j}} \dot{\psi}_i \dot{P}_{\bar{j}}] \omega_{\psi}^n \\ (8.3) \quad &+ \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \Delta_{\psi} \dot{\psi} \omega_{\psi}^n. \end{aligned}$$

Differentiating the second equation in (7.1) we get

$$g_{\psi}^{i\bar{j}} \dot{P}_{i\bar{j}} = \dot{g}_{\psi}^{i\bar{j}} (-P_{i\bar{j}} + R_{i\bar{j}}(\omega)) = f^{i\bar{j}} (-P_{i\bar{j}} + R_{i\bar{j}}(\omega)).$$

So using (3.5) and the fact that $-P_{i\bar{j}} + R_{i\bar{j}}(\omega)$ is a harmonic form, we have

$$\begin{aligned} - \int_M g_{\psi}^{i\bar{j}} \dot{P}_i \dot{\psi}_{\bar{j}} \omega_{\psi}^n &= \int_M \Delta_{\psi} \dot{P} \dot{\psi} \omega_{\psi}^n = \int_M (f^{i\bar{j}} (-P_{i\bar{j}} + R_{i\bar{j}}(\omega))) \dot{\psi} \omega_{\psi}^n \\ (8.4) \quad &= \int_M ((-P^{i\bar{j}} + R^{i\bar{j}}(\omega))) \dot{\psi}_i \dot{\psi}_{\bar{j}} \omega_{\psi}^n. \end{aligned}$$

Combining (8.3), (8.4) with the following equations

$$f_{i\bar{j}} = P_{i\bar{j}} - h_{i\bar{j}} \text{ and } \Delta_{\psi} \frac{\partial}{\partial t} \psi = -\Delta_{\psi} f = -(S - \underline{S}),$$

we obtain

$$\begin{aligned} \frac{d}{dt} \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_{\psi}^n &= -\frac{1}{V} \int_M |\dot{\psi}_{i\bar{j}}|^2 \omega_{\psi}^n - \frac{1}{V} \int_M |\dot{\psi}_{i\bar{j}}|^2 \omega_{\psi}^n \\ &+ \frac{1}{V} \int_M (-P^{i\bar{j}} + R^{i\bar{j}}(\omega)) \dot{\psi}_i \dot{\psi}_{\bar{j}} \omega_{\psi}^n - \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 (S - \underline{S}) \omega_{\psi}^n. \end{aligned}$$

Last we insert the Ricci identity, i.e.

$$\int_M |\dot{\psi}_{i\bar{j}}|^2 \omega_{\psi}^n = \int_M |\dot{\psi}_{i\bar{j}}|^2 \omega_{\psi}^n + \int_M R^{i\bar{j}} \dot{\psi}_i \dot{\psi}_{\bar{j}} \omega_{\psi}^n$$

into the above differential inequality and obtain immediately,

$$\begin{aligned}
& \frac{d}{dt} \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_\psi^n \\
&= -2 \frac{1}{V} \int_M |\nabla \nabla \dot{\psi}|^2 \omega_\psi^n + \frac{1}{V} \int_M (-P^{i\bar{j}} - R^{i\bar{j}} + R^{i\bar{j}}(\omega)) \dot{\psi}_i \dot{\psi}_{\bar{j}} \omega_\psi^n \\
&\quad - \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 (S - \underline{S}) \omega_\psi^n \\
&= -2 \frac{1}{V} \int_M (L_t \dot{\psi}, \dot{\psi}) \omega_\psi^n - \frac{1}{V} \int_M f^{i\bar{j}} \dot{\psi}_i \dot{\psi}_{\bar{j}} \omega_\psi^n - \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 (S - \underline{S}) \omega_\psi^n.
\end{aligned}$$

□

Since g_ψ converges to a cscK metric, we have that both f and $S - \underline{S}$ tend to zero. As a result we deduce that for any small ϵ ,

$$(8.5) \quad \frac{d}{dt} \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_\psi^n \leq -2 \frac{1}{V} \int_M (L_t \dot{\psi}, \dot{\psi}) \omega_\psi^n + \epsilon \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_\psi^n.$$

Moreover, the Futaki invariant implies that

$$\begin{aligned}
(8.6) \quad 0 &= F(X) = \int_M X(f) \omega_\psi^n = \int_M (\theta_X(\psi))_i f_{\bar{i}} \omega_\psi^n \\
&= - \int_M \theta_X(\psi) (S - \underline{S}) \omega_\psi^n = - \int_M (\theta_X(\psi))_i \dot{\psi}_{\bar{i}} \omega_\psi^n
\end{aligned}$$

for any $X = \uparrow \bar{\partial} \theta_X \in \eta(M)$.

Let L_t be the Lichnerowicz operator. It is a positive semidefinite, self-adjoint operator and $L_t \psi = 0$ if and only if $\nabla \psi = \uparrow \bar{\partial} \psi$ is a holomorphic vector field. We are going to obtain the first eigenvalue of L_t with regard to the metric $g_{\phi(t)}$ first. We define the set

$$\begin{aligned}
A_t &= \{f \in C_R^\infty(M) \mid \int_M f \omega_\phi^n = 0; \\
&\quad \int_M (\theta_X(\phi))_i f_{\bar{i}} \omega_\phi^n = 0, \forall X = \uparrow \bar{\partial} \theta_X(\phi) \in \eta(M)\}.
\end{aligned}$$

Then (8.6) implies $f_\phi \in A_t$ for $\Delta_\phi f_\phi = S_\phi - \underline{S}$. We further define

$$\begin{aligned}
\lambda(\phi) &= \inf_{f \in A_t} \{c \mid \int_M |\nabla \nabla f|^2 \omega_\phi^n \geq c \int_M |\nabla f|^2 \omega_\phi^n\} \\
&= \inf_{f \in A_t} \{c \mid c \leq \int_M |\nabla \nabla f|^2 \omega_\phi^n; \int_M |\nabla f|^2 \omega_\phi^n = 1\}.
\end{aligned}$$

We prove a similar lemma to Chen-Li-Wang [19].

Lemma 8.2. *We have the uniform lower bound of the Lichnerowicz operator L_t i.e.*

$$\lambda > 0, \forall t \geq 0.$$

Proof. We prove the lemma by the contradiction method. If the conclusion fails, then we can choose a sequence t_s such that

$$\begin{aligned}
& \int_M |\nabla_s \nabla_s f_s|^2 \omega_{\phi_s}^n < \lambda(\phi_s) \int_M |\nabla_s f_s|^2 \omega_{\phi_s}^n = \frac{1}{s} \int_M |\nabla_s f_s|^2 \omega_{\phi_s}^n, \int_M f_s \omega_{\phi_s}^n = 0 \\
& \text{and } \int_M (\theta_X(\phi_s))_i (f_s)_{\bar{i}} \omega_{\phi_s}^n = 0, \forall X = \uparrow \bar{\partial} \theta_X(\phi_s) \in \eta(M).
\end{aligned}$$

We further scale f_s such that

$$(8.7) \quad \int_M |\nabla_s \nabla_s f_s|^2 \omega_{\phi_s}^n < \frac{1}{s}, \int_M |\nabla f_s|^2 \omega_{\phi_s}^n = 1, \int_M f_s \omega_{\phi_s}^n = 0$$

and $\int_M (\theta_X(\phi_s))_i (f_s)_{\bar{i}} \omega_{\phi_s}^n = 0, \forall X \Rightarrow \bar{\partial} \theta_X(\phi_s) \in \eta(M).$

Recall that Theorem 7.8 gives a subsequence of g_{ϕ_s} satisfying

$$\lim_{j \rightarrow \infty} g(\phi(t_{s_j})) = g(\phi_\infty) \text{ in } C^l \text{ for any } l \geq 0.$$

Moreover, combining Ricci identity

$$\int_M |(f_s)_{i\bar{j}}|^2 \omega_{\phi_s}^n = \int_M |(f_s)_{ij}|^2 \omega_{\phi_s}^n + \int_M R^{i\bar{j}} (f_s)_i (f_s)_{\bar{j}} \omega_{\phi_s}^n$$

and by the uniform boundness of Ricci curvature we obtain

$$\int_M |\nabla \bar{\nabla} f_s|^2 \omega_{\phi_s}^n \leq \frac{1}{s} + C(Ric).$$

This inequality together with (8.7) and the Poincaré inequality provides a $W^{2,2}$ -weak convergent subsequence of f_s such that

$$f_s \rightharpoonup f_\infty \text{ in } W^{2,2}.$$

In addition, the Sobolev imbedding theorem implies

$$\nabla_s f_s \rightarrow \nabla_\infty f_\infty \text{ in } L^2 \text{ and } f_s \rightarrow f_\infty \text{ in } L^2.$$

Accordingly, by the assumption we have

$$(8.8) \quad \begin{aligned} \int_M |\nabla_\infty \nabla_\infty f_\infty|^2 \omega_\infty^n &\leq \liminf_{s \rightarrow \infty} \int_M |\nabla_\infty \nabla_\infty f_s|^2 \omega_{\phi_s}^n \\ &\leq C \liminf_{s \rightarrow \infty} \int_M |\nabla_s \nabla_s f_s|^2 \omega_{\phi_s}^n = 0, \\ \lim_{s \rightarrow \infty} \int_M |\nabla_s f_s|^2 \omega_{\phi_s}^n &= \int_M |\nabla_\infty f_\infty|^2 \omega_\infty^n = 1, \\ \lim_{s \rightarrow \infty} \int_M f_s \omega_{\phi_s}^n &= \int_M f_\infty \omega_\infty^n = 0. \end{aligned}$$

On the other hand we turn to the Futaki invariant,

$$0 = \int_M X(f_s) \omega_{\phi_s}^n = \int_M (\theta_X(\phi_s))_i (f_s)_{\bar{i}} \omega_{\phi_s}^n = 0$$

for any $X \in \eta(M, g_\phi)$ which implies

$$(8.9) \quad 0 = \int_M X(f_\infty) \omega_\infty^n = \int_M \theta_X(\phi_\infty)_i f_{\infty \bar{i}} \omega_\infty^n.$$

after taking limit. Notice that the complex structure is fixed, then we have

$$\eta(M, g_\infty) = \eta(M, g(\phi_s)).$$

Together with (8.9) it implies f_∞ does not belong to

$$Ker L_\infty = \{\theta_X(\phi_\infty) | X \Rightarrow \bar{\partial} \theta_X(\phi_\infty) \in \eta(M, g_\infty); \int_M \theta_X(\phi_\infty) \omega_{\phi_\infty}^n = 0\}.$$

Consequently, we have

$$(8.10) \quad \int_M |\nabla_\infty \nabla_\infty f_\infty|^2 \omega_\infty^n > \lambda \int_M |\nabla_\infty f_\infty|^2 \omega_\infty^n = \lambda > 0.$$

That contradicts to (8.8) and the lemma follows. \square

Since the eigenvalue λ is invariant under the holomorphic transformation, we obtain the uniform positive lower bound of the first eigenvalue of $g(\psi(t))$. Therefore Lemma 8.2 immediately implies

$$\int_M \dot{\psi} L_t \dot{\psi} \omega_\psi^n \geq \lambda \int_M |\dot{\psi}|^2 \omega_\psi^n.$$

Substituting this inequality into (8.5), by the Gronwall's inequality we obtain the exponential decay of the energy μ_1 ,

$$(8.11) \quad \mu_1(t) = \frac{1}{V} \int_M |\nabla \dot{\psi}|^2 \omega_\psi^n \leq \mu_1(0) e^{-\theta t}.$$

Moreover the inequality (8.11) together with the Poincaré inequality and the normalization condition

$$\int_M \partial_t \varphi \omega_\varphi^n = 0,$$

implies

$$(8.12) \quad \mu_0(t) = \frac{1}{V} \int_M |\dot{\psi}|^2 \omega_\psi^n \leq \mu_0(0) e^{-\theta t}.$$

Furthermore, we can control the evolution of $\mu_l(t) = \frac{1}{V} \int_M |\nabla^l \dot{\psi}|^2 \omega_\psi^n$ by the following lemma.

Lemma 8.3. *For any $l \geq 2$ the following inequality holds*

$$\partial_t \mu_l(t) \leq C \mu_0(t).$$

Proof. We prove this lemma in real coordinates. Let $I = (i_1, \dots, i_l)$, $J = (j_1, \dots, j_l)$ and $g^{IJ} = g^{i_1 j_1} \dots g^{i_l j_l}$. Differentiating $|\nabla^l \dot{\psi}|^2$ with respect to t and using (8.2), we get

$$\begin{aligned} \frac{d}{dt} |\nabla^l \dot{\psi}|^2 &= \sum g^{i_1 j_1} \dots f^{pq} \dots g^{i_l j_l} \dot{\psi}_{i_1, \dots, p, \dots, i_l} \dot{\psi}_{j_1, \dots, q, \dots, j_l} \\ &\quad + g_\psi^{IJ} \ddot{\psi}_I \dot{\psi}_J + g_\psi^{IJ} \dot{\psi}_I \ddot{\psi}_J \\ &= \sum g^{i_1 j_1} \dots f^{pq} \dots g^{i_l j_l} \dot{\psi}_{i_1, \dots, p, \dots, i_l} \dot{\psi}_{j_1, \dots, q, \dots, j_l} \\ &\quad + g_\psi^{IJ} (\Delta_\psi \dot{\psi})_I \dot{\psi}_J - g_\psi^{IJ} \dot{P}_I \dot{\psi}_J + g_\psi^{IJ} \dot{\psi}_I (\Delta_\psi \dot{\psi})_J - g_\psi^{IJ} \dot{\psi}_I \dot{P}_J. \end{aligned}$$

Meanwhile, a standard computation gives

$$\Delta_\psi |\nabla^l \dot{\psi}|^2 = Rm * \nabla^l \dot{\psi} * \nabla^l \dot{\psi} + g_\psi^{IJ} (\Delta_\psi \dot{\psi})_I \dot{\psi}_J + g_\psi^{IJ} \dot{\psi}_I (\Delta_\psi \dot{\psi})_J + |\nabla^{l+1} \dot{\psi}|^2.$$

Combining these two equalities, we obtain

$$\begin{aligned}
\partial_t \frac{1}{V} \int_M |\nabla^l \dot{\psi}|^2 \omega_\psi^n &= \frac{1}{V} \int_M [\sum g^{i_1 j_1} \dots f^{p q} \dots g^{i_l j_l} \dot{\psi}_{i_1, \dots, p, \dots, i_l} \dot{\psi}_{j_1, \dots, q, \dots, j_l} \\
&\quad - g_\psi^{I J} \dot{P}_I \dot{\psi}_J - g_\psi^{I J} \dot{\psi}_I \dot{P}_J - Rm * \nabla^l \dot{\psi} * \nabla^l \dot{\psi} - |\nabla^{l+1} \dot{\psi}|^2] \omega_\psi^n \\
&\quad + \frac{1}{V} \int_M |\nabla^l \dot{\psi}|^2 \Delta_\psi \dot{\psi} \omega_\psi^n \\
&\leq C \mu_l(t) - \mu_{l+1}(t) \\
&\leq C(\epsilon) \mu_1(t) + (C\epsilon - 1) \mu_{l+1}(t).
\end{aligned}$$

The last inequality holds by the interpolation inequality. Then the lemma follows by (8.11) when we choose ϵ to be small enough. \square

Together with (8.12) Lemma 8.3 implies

$$\frac{1}{V} \int_M |\nabla^l \dot{\psi}|^2 \omega_\psi^n \leq \mu_l(0) e^{-\theta t}$$

for $l \geq 0$ by the Gronwall's inequality again. Since all $\varrho_t^* \omega_{\psi_t}$ are equivalent and the Sobolev constant and the Poincaré constant are invariant under the holomorphic transformations, we deduce that ω_{ψ_t} has uniform Sobolev constant and Poincaré constant. The Sobolev imbedding theorem implies

$$(8.13) \quad |\dot{\psi}|_{C^l(g_\psi)} \leq C e^{-\theta t}.$$

In view of the identity

$$\psi(t) = \psi(0) + \int_0^t \dot{\psi} dt,$$

we have

$$|\psi(t)|_{C^0} \leq |\psi(0)|_{C^0} + C e^{-\theta t}.$$

Now note that the potential and the Ricci curvature are uniformly bounded

$$C_1 \omega_\psi \leq Ric_\psi \leq C_2 \omega_\psi.$$

According to Chen-He's compactness theorem [18], we obtain $|\psi|_{2, \alpha}$ is uniformly bounded. Moreover the higher order bound follows from Theorem 5.38. Consequently, (8.13) gives rise to

$$|\psi - \psi_\infty|_{C^l(g_\infty)} \leq C_l e^{-\theta t}.$$

Hence combining with Theorem 7.3 and Theorem 7.8, we obtain the following stability theorem.

Theorem 8.4. *For each Kähler class which admits a cscK metric ω , there exists a small constant ϵ_0 such that, if $|\varphi_0|_{C^{2, \alpha}(M)} \leq \epsilon_0$, then the solution to the pseudo-Calabi flow exists for all times and converges exponentially fast to a unique cscK metric nearby in the Kähler class $[\omega]$.*

An immediate corollary is

Corollary 8.5. *If the pseudo-Calabi flow converges to a cscK metric by passing to a subsequence if necessary, then the limit cscK metric (depends on the subsequence) must be unique.*

Remark 8.1. We hope our method of proving the stability problem may be useful in other geometric flow problems. Since the pseudo-Calabi flow is just the Kähler-Ricci flow in the canonical class. By using the similar technique of the pseudo-Calabi flow, we shall prove the stability of the Kähler-Ricci flow near a Kähler-Ricci soliton in the subsequent paper [56]. Meanwhile, we prove the stability of the Calabi flow near an extremal metric in [38].

9. FURTHER REMARKS AND PROBLEMS

A Chen's conjecture [14] says that

Conjecture 9.1. (*Chen [14]*) *A global $C^{1,1}$ K -energy minimizer in any Kähler class must be smooth.*

This conjecture has been proved in the canonical Kähler class via the weak Kähler-Ricci flow [15][22][23][51]. In general Kähler class, the conjecture will be verified if one proves the following question:

Question 9.2. (*Chen [13]*) *Can the pseudo-Calabi flow start from a $C^{1,1}$ Kähler potential?*

In Subsection 5.5, we have obtained a partial estimate related to this conjecture. In the subsequent paper, we will relax the regularity of the initial Kähler potential.

As we have proved, the pseudo-Calabi flow shares some properties with the Calabi flow; recall for instance Theorem 2.2. However, even in the canonical class $C_1(M)$, they are extremely different. Within $C_1(M)$ the pseudo-Calabi flow is precisely the Kähler-Ricci flow. The Kähler-Ricci flow has many properties such as the long time existence [8] and the Perelman's estimate [47]. Chen-Tian proved the convergence of the Kähler-Ricci flow on Kähler-Einstein manifolds [20][21]. Later Zhu [53] extended their theorems to the Kähler manifolds which admit a Kähler-Ricci soliton. One may ask if the pseudo-Calabi flow inherits these fine properties of the Kähler-Ricci flow.

Conjecture 9.3. (*Chen [13]*) *The pseudo-Calabi flow has long time existence.*

There is a conjecture in [14] saying

Conjecture 9.4. (*Chen [14]*) *The existence of cscK metrics implies that the K -energy is proper in the sense that it bounds the geodesic distance.*

Inspired from this conjecture, one may generalize Theorem 2.3 to the following question.

Question 9.5. (*Chen [13]*) *On Kähler manifolds admit a cscK metric, does the pseudo-Calabi flow converge to a cscK metric?*

Note that Lemma 7.4 provides a method to control the norm of the potential by its geodesic distance in the critical submanifold.

Recently, Donaldson [31][30][29][28] proved the following theorem.

Theorem 9.6. (*Donaldson [31]*) *If a polarized complex toric surface with zero Futaki invariant is K -stable, then it admits a cscK metric.*

Following the same line of Donaldson's theorem, one may ask

Question 9.7. (*Chen [13]*) Let M be a polarized complex toric surface with zero Futaki invariant. If M is K -stable, does the pseudo-Calabi flow converge to a cscK metric up to holomorphic diffeomorphisms?

Actually, according to Theorem 2.3 the convergence up to holomorphic diffeomorphisms in Question 9.7 implies the exponential convergence.

REFERENCES

- [1] Sigurd B. Angenent. Nonlinear analytic semiflows. *Proc. Roy. Soc. Edinburgh Sect. A*, 115(1-2):91–107, 1990.
- [2] Melvin S. Berger. *Nonlinearity and functional analysis*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977. Lectures on nonlinear problems in mathematical analysis, Pure and Applied Mathematics.
- [3] Eugenio Calabi. Extremal Kähler metrics. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 259–290. Princeton Univ. Press, Princeton, N.J., 1982.
- [4] Eugenio Calabi. Extremal Kähler metrics. II. In *Differential geometry and complex analysis*, pages 95–114. Springer, Berlin, 1985.
- [5] Eugenio Calabi and Xiuxiong Chen. Private communication.
- [6] Eugenio Calabi and Xiuxiong Chen. The space of Kähler metrics. II. *J. Differential Geom.*, 61(2):173–193, 2002.
- [7] Simone Calamai. The Calabi’s metric for the space of Kähler metrics. arXiv:1004.5482v1, 2010.
- [8] Huai Dong Cao. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81(2):359–372, 1985.
- [9] Isaac Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.
- [10] Xiuxiong Chen. On the lower bound of the Mabuchi energy and its application. *Internat. Math. Res. Notices*, (12):607–623, 2000.
- [11] Xiuxiong Chen. The space of Kähler metrics. *J. Differential Geom.*, 56(2):189–234, 2000.
- [12] Xiuxiong Chen. Calabi flow in Riemann surfaces revisited: a new point of view. *Internat. Math. Res. Notices*, (6):275–297, 2001.
- [13] Xiuxiong Chen. Private communication. 2007.
- [14] Xiuxiong Chen. Space of Kähler metrics (IV)—On the lower bound of the K-energy. arXiv:0809.4081v2, 2008.
- [15] Xiuxiong Chen and Weiyue Ding. Ricci flow on surfaces with degenerate initial metrics. *J. Partial Differential Equations*, 20(3):193–202, 2007.
- [16] Xiuxiong Chen and Weiyong He. The Calabi flow on Kähler surface with bounded sobolev constant—I. arXiv:0710.5159, 2007.
- [17] Xiuxiong Chen and Weiyong He. The Calabi flow on toric Fano surface. arXiv:0807.3984, 2008.
- [18] Xiuxiong Chen and Weiyong He. On the Calabi flow. *Amer. J. Math.*, 130(2):539–570, 2008.
- [19] Xiuxiong Chen, Haozhao Li, and Bing Wang. Kähler-Ricci flow with small initial energy. *Geom. Funct. Anal.*, 18(5):1525–1563, 2009.
- [20] Xiuxiong Chen and Gang Tian. Ricci flow on Kähler-Einstein surfaces. *Invent. Math.*, 147(3):487–544, 2002.
- [21] Xiuxiong Chen and Gang Tian. Ricci flow on Kähler-Einstein manifolds. *Duke Math. J.*, 131(1):17–73, 2006.
- [22] Xiuxiong Chen and Gang Tian. Geometry of Kähler metrics and foliations by holomorphic discs. *Publ. Math. Inst. Hautes Études Sci.*, (107):1–107, 2008.
- [23] Xiuxiong Chen, Gang Tian, and Zhou Zhang. On the weak Kähler-Ricci flow. 2008. arXiv:0802.0809.
- [24] Piotr T. Chruściel. Semi-global existence and convergence of solutions of the Robinson-Trautman (2-dimensional Calabi) equation. *Comm. Math. Phys.*, 137(2):289–313, 1991.
- [25] Giuseppe Da Prato and Pierre Grisvard. Equations d’évolution abstraites non linéaires de type parabolique. *Ann. Mat. Pura Appl. (4)*, 120:329–396, 1979.
- [26] Wei Yue Ding. An introduction to harmonic maps. unpublished.

- [27] Simon K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 13–33. Amer. Math. Soc., Providence, RI, 1999.
- [28] Simon K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differential Geom.*, 62(2):289–349, 2002.
- [29] Simon K. Donaldson. Interior estimates for solutions of Abreu’s equation. *Collect. Math.*, 56(2):103–142, 2005.
- [30] Simon K. Donaldson. Extremal metrics on toric surfaces: a continuity method. *J. Differential Geom.*, 79(3):389–432, 2008.
- [31] Simon K. Donaldson. Constant scalar curvature metrics on toric surfaces. *Geom. Funct. Anal.*, 19(1):83–136, 2009.
- [32] James Eells, Jr. and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.
- [33] Lawrence C. Evans. Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Comm. Pure Appl. Math.*, 35(3):333–363, 1982.
- [34] Avner Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.
- [35] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [36] Daniel Guan. Extremal solitons and exponential C^∞ convergence of the modified Calabi flow on certain \mathbb{CP}^1 bundles. *Pacific J. Math.*, 233(1):91–124, 2007.
- [37] Richard S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N.S.)*, 7(1):65–222, 1982.
- [38] Hongnian Huang and Kai Zheng. Stability of calabi flow near extremal metric. Submitted, 2009.
- [39] T. Kato. *Abstract differential equations and nonlinear mixed problems*. Lezioni Fermiane. [Fermi Lectures]. Scuola Normale Superiore, Pisa, 1985.
- [40] S. N. Kružkov, A. Kastro, and M. Lopes. Schauder type estimates, and theorems on the existence of the solution of fundamental problems for linear and nonlinear parabolic equations. *Dokl. Akad. Nauk SSSR*, 220:277–280, 1975.
- [41] N. V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(3):487–523, 670, 1982.
- [42] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.
- [43] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
- [44] Alessandra Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995.
- [45] Toshiki Mabuchi. K -energy maps integrating Futaki invariants. *Tohoku Math. J. (2)*, 38(4):575–593, 1986.
- [46] Toshiki Mabuchi. Some symplectic geometry on compact Kähler manifolds. I. *Osaka J. Math.*, 24(2):227–252, 1987.
- [47] Grisha Perelman. Unpublished note on Kähler-Ricci flow.
- [48] Richard Schoen and Shing-Tung Yau. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu, Translated from the Chinese by Ding and S. Y. Cheng, Preface translated from the Chinese by Kaising Tso.
- [49] Stephen Semmes. Complex Monge-Ampère and symplectic manifolds. *Amer. J. Math.*, 114(3):495–550, 1992.
- [50] Santiago R. Simanca. Heat flows for extremal Kähler metrics. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(2):187–217, 2005.
- [51] Jian Song and Gang Tian. The Kähler-ricci flow through singularities. Unpublished, 2009.
- [52] Gang Tian. *Canonical metrics in Kähler geometry*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld.

- [53] Gang Tian and Xiaohua Zhu. Convergence of Kähler-Ricci flow. *J. Amer. Math. Soc.*, 20(3):675–699 (electronic), 2007.
- [54] Valentino Tosatti and Ben Weinkove. The Calabi flow with small initial energy. *Math. Res. Lett.*, 14(6):1033–1039, 2007.
- [55] Shing-Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.
- [56] Kai Zheng. Stability of Kähler-Ricci flow in the space of Kähler metrics. arXiv:1004.2695, 2010.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706, USA
E-mail address: `xiuxiong.chen@gmail.com`

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCES, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, P.R. CHINA
E-mail address: `kaizheng@amss.ac.cn`